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CONTENTS

FOREWORD

The Brief Report of **19th International Conference on Statistical Sciences** shows in it the interaction that took place among researchers, scientists, statisticians, mathematicians, the experts of management sciences, medical sciences, etc. etc. It speaks less whereas the amount of work done during these three days virtually and physically cannot be written with more brevity. Obviously, there are papers presented, discussions held and the knowledge shared during this Conference. This exercise is of great importance for the sake of knowledge and for the sake of policy making.

The virtuality in this Conference added after COVID-19 is also one of the hallmarks which include the scientists like Professor Dr. Shahjahan Khan, (former President ISOSS), University of Southern Queensland, Australia on "Statistical Models in Meta-Analysis with Applications in Health Sciences" followed by Dr. Noor Afzan Binti Salleh from Malaysia on "Artificial Intelligence Benefits and Challenges for Business: Systematic Literature Review". A few others also added to the body of knowledge through their virtual presence.

The tribute paid to Late Professor Dr. Munir Ahmad, Founding President of Islamic Countries Society of Statistical Sciences gives the audience an air of his presence and serves as a chord to interlink these knowledge sharing activities to help take the flag yonder than the shore of time. This ever presence was, is, and will remain part of all the activities that ISOSS will plan, implement and evaluate.

Professor Dr. Mumtaz Muhammad Khan
Vice-President ISOSS and
Rector, Lahore Institute of Science & Technology

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KEY NOTE SPEECH

STATISTICAL MODELS IN META-ANALYSIS WITH APPLICATIONS IN HEALTH SCIENCES

Shahjahan Khan[1,2] and **Muhammed Ashraf Memon**[2]
[1] School of Sciences and Engineering, Asian University of Bangladesh, Ashulia, Dhaka 1341, Bangladesh
[2] School of Mathematics, Physics and Computing
Centre of Health Research, University of Southern Queensland
Toowoomba, Australia. Email: Shahjahan.Khan@usq.edu.au

ABSTRACT

Although originated in education (Glass, 1976) and widely used in health sciences, meta-analysis is now being used to estimate the common effect size in almost all disciplines. In the age of evidence-based medicine meta-analysis has been increasingly used to synthesize the findings of multiple independent studies on the same topic often as part of systematic reviews and with conflicting outcomes. Meta-analysis refers to the statistical analysis of summary results from independent primary studies focused on the same intervention or treatment effect to decide on policies, or clinical practices, or programs. The main objective of meta-analysis is to find an estimate of the common effect size using data from all relevant independent studies and present the results in a forest plot representing confidence intervals along with some key summary statistics. Like many other statistical methods, heterogeneity is an important issue in meta-analysis, which requires careful consideration. Different statistical models use different weighting system to estimate the common effect size and find the standard error of the estimator to calculating confidence intervals. As covered in (S Khan, 2020), this talk presents meta-analysis as a key component of systematic review, highlights selected effect size measures, discusses different statistical models, and illustrates meta-analyses using commonly used statistical models for several data sets from health sciences.

1. INTRODUCTION

Meta-Analysis is being increasingly used in many evidence-based decision-making in a wide range of disciplines and in many governmental and commercial organisations. The statistical method is being frequently used in health and medical research to determine clinical practices and health policies. (Memon, Khan, Alam, Rahman, & Yunus, 2021) covers details on the systematic review with its pros and cons.

The validity of evidence, and hence the decisions based on them, are reliant on the quality of the data, the methodology employed to extract them, the robustness and totality of the evidence and relevant methods to analyse them. Hence, the decision-makers needs to be aware of the factors that impact the quality of evidence along with any shortcomings in the process of gathering and processing such evidence. For further details please see (S. Khan, Memon, & Memon, 2019).

Meta-analysis is a statistical method used to combine numerical summary results on effect size measures, extracted during the systematic review process from independent studies to synthesize a pooled result. The synthesis of a summary statistic from aggregate data in all available independent trials/studies is achieved by pooling them to find an estimate of the unknown population effect size. Meta-analysis enables us to arrive at a better estimate of the population effect size compared to that reported in individual studies especially when results of independent studies are conflicting.

There are a number of statistical models that are commonly used in meta-analyses. See chapter 2 of (S Khan, 2020) for different relevant statistical aspects of meta-analysis.

2. STATISTICAL MODELS

The main feature of any meta-analysis is the forest plot. It is a scatter plot of the 95% confidence intervals of effect size for different independent primary studies included in the meta-analysis as well as that of the common effect size obtained by pooling the results from all studies. The confidence intervals are calculated using the point estimate and standard error of the estimator of the effect size. Prior to the calculation of the point estimate and standard error of the estimator of the effect size the calculation of weights for each of the studies is essential.

Different statistical models use different formula to find the study weights. In fact, the main focus of every statistical model is to re-distribute the weights appropriately in the meta-analysis. The main objective of re-distribution of weights is to find the most precise estimate.

If the studies are not heterogeneous the fixed effect model is fine. Unfortunately, in real-life most of the studies are heterogeneous and hence fixed effect model is not appropriate. As part of the process test of heterogeneity is essential. This is done by using Cochrane's Q statistics which follows a Chi-squared distribution. If the test outcome is significant the fixed effect model is inappropriate and meta-analysis requires appropriate statistical model.

Every meta-analysis assumes that there is a common effect across all studies, and pulls data from all studies to estimate the unknown common effect size, usually denoted by θ, using an appropriate statistical model. The commonly used statistical models are briefly discussed below.

(a) The Fixed Effect Model

The fixed effect (FE) model represents the observed effect size for any study *i* by $\hat{\theta}_i = \theta + \varepsilon_i$, where error term ε_i is the difference between the common true effect size, θ and the observed sample effect size for study *i*, $\hat{\theta}_i$. It is assumed that the error are independent and follow normal distribution with mean 0 and variance σ_i^2, that is, $\varepsilon_i \sim N(0, \sigma_i^2)$ for $i = 1,2, \ldots, k$, where *k* is the number of studies included in the meta-analysis. The fixed effect model assumes that there is only one source of variation, the within-study variation, which is the estimation error ε_i. The variance of the estimation error for the ith study, σ_i^2 is estimated by $\hat{\sigma}_i^2 = v_i$, the sample variance.

Under the FE model, the common effect size θ is estimated by using the inverse variance weight, $\hat{\theta}_{FE} = \sum_{i=1}^{k} w_i \hat{\theta}_i \Big/ \sum_{i=1}^{k} w_i$, where $w_i = \frac{1}{v_i}$, the weight for the ith study in which v_i is the sample variance. The estimated variance of the estimator of θ is given by $Var(\hat{\theta}_{FE}) = \left(\sum_{i=1}^{k} \frac{1}{v_i} \right)^{-1}$ and the standard error is defined as $SE(\hat{\theta}_{FE}) = \sqrt{\left(\sum_{i=1}^{k} \frac{1}{v_i} \right)^{-1}}$. The confidence interval and test of hypothesis for the effect size θ under the FE model are based on the above point estimate, $\hat{\theta}_{FE}$ and standard error, $SE(\hat{\theta}_{FE})$.

(b) The Random Effects Model

The random effects (REs) model was proposed by (DerSimonian & Laird, 1986). The REs model assumes that the true effect size is not identical for every study but have enough in common to justify the synthesis of results to produce an "average effect size". The REs model re-distributes the weights using two (within and between study) sources of variation. Under the REs model, the observed effect size for any study *i* is represented by $\hat{\theta}_i = \theta + \zeta_i + \varepsilon_i$, where $\zeta_i = \theta_i - \theta$ and $\varepsilon_i = \hat{\theta}_i - \theta_i$ represent the two sources of variation with the assumption that both follow normal distribution, $\zeta_i \sim N(0, \tau^2)$ and $\varepsilon_i \sim N(0, \sigma_i^2)$ for $i = 1,2, \ldots, k$. The sample estimates of τ^2 is $\hat{\tau}^2$. The combined variance of study *i*, that is, $\sigma_i^{2*} = \left(\sigma_i^2 + \tau^2\right)$ is estimated by $v_i^* = \left(v_i + \hat{\tau}^2\right)$, and then the weight for study *i* under the REs model becomes $w_i^* = \left(v_i + \hat{\tau}^2\right)^{-1}$. The common effect size estimator under the REs model is $\hat{\theta}_{RE} = \frac{\sum_{i=1}^{k} w_i^* \hat{\theta}_i}{\sum_{i=1}^{k} w_i^*}$, and the standard error of the estimator is $SE(\hat{\theta}_{RE}) = \sqrt{\left(\sum_{i=1}^{k} w_i^* \right)^{-1}}$. The confidence interval and test of hypothesis for the effect size θ under the REs model are based on the above point estimate, $\hat{\theta}_{RE}$ and standard error, $SE(\hat{\theta}_{RE})$.

(c) The Inverse Variance Heterogeneity Model

The inverse variance heterogeneity (IVhet) model was introduced by (S.A. Doi, Barendregt, Khan, Thalib, & Williams, 2015a). It emphasises that the fixed effect model based estimator variance can be made closer to the observed variance by modelling over-dispersion through a quasi-likelihood approach (see (Kulinskaya & Olkin, 2014)). This implies that the meta-analysis is performed under a fixed effect assumption $(\tau^2 = 0)$ and the variance of the estimator is inflated to account for the heterogeneity.

The IVhet model estimates the common effect size θ by $\hat{\theta}_{IVhet} = \sum_{i=1}^{k} w_i \hat{\theta}_i$ and the standard error of the estimator is

$$SE(\hat{\theta}_{IVhet}) = \sqrt{\sum_{i=1}^{k} \left[\left(\frac{1}{v_i} \Big/ \sum_{i=1}^{k} \frac{1}{v_i} \right)^2 (v_i + \hat{\tau}^2) \right]}.$$

The confidence interval and test of hypothesis for the effect size θ under the IVhet model are based on the above point estimate, $\hat{\theta}_{IVhet}$ and standard error, $SE(\hat{\theta}_{IVhet})$.

Through extensive simulations (S.A. Doi, Barendregt, Khan, Thalib, & Williams, 2015c) showed that the IVhet model performs better than the REs model under varieties of conditions.

Issues with Random Effects Model Meta-Analysis

The random effects (REs) model (unjustifiably) assumes that the studies in the meta-analysis is a random sample of all population studies. In reality, this hardly ever occurs.

The REs assumes that the within study errors are normally distributed. It is also assume that effect size in each study is a random variable which follows a normal distribution.

Unfortunately, hardly the validity of the assumptions is rarely checked, effecting the results.

Under REs model, the coverage (probability) of the confidence interval drops significantly with increasing of heterogeneity. So, it fails to maintain nominal confidence level.

All meta-analysis model redistributes weight to calculate the pooled estimate of the common effect size. REs model allocates more weights to the small studies (and therefore less weights to larger studies) with increasing heterogeneity. Allocating more weight to smaller studies is unjustifiable and wrong, and leads to poor pooled estimate.

In the presence of larger heterogeneity, the individual study weights under the REs model become equal and thus the REs model estimator returns out to be an arithmetic mean of the individual effect sizes rather than the intended weighted average.

The inverse variance heterogeneity (IVhet) model does not suffer from the above side-effect of the REs model. Thus it differs from the REs model estimate in two ways: (1) larger trials contribute more (weights) to the pooled estimates (as opposed to penalising larger trials in the REs model), and (2) yields confidence intervals that maintain the nominal coverage (probability) under uncertainty due to increased heterogeneity. In addition, no distributional assumptions on the estimator of the effect size is required for the IVhet model, and hence it provides a robust statistical method compared to the REs model.

Advantages of IVhet Model over REs Model

Doi (S. A. Doi et al., 2015a) states that the IVhet model has clear advantages over the REs model as it resolves the two main problems of the latter model. The first advantage of the IVhet model is that the coverage (probability) of the confidence interval remains at the nominal (usually 95%) level, unlike that of the REs model for which the coverage drops significantly with increasing heterogeneity. The second advantage is that the IVhet model maintains the inverse variance weights of individual studies, unlike the REs model which allocates more weights to the small studies (and therefore less weights to larger studies) with increasing heterogeneity.

In the presence of larger heterogeneity, the individual study weights under the REs model become equal and thus the REs model estimator returns out to be an arithmetic mean of the individual effect sizes rather than the intended weighted average. The IVhet model does not suffer from this side-effect of the REs model. Thus, it differs from the REs model estimate in two ways: (1) larger trials contribute more (weights) to the pooled estimates (as opposed to penalising larger trials in the REs model), and (2) yields confidence intervals that maintain the nominal coverage (probability) under uncertainty due to increased heterogeneity. In addition, no distributional assumptions on the estimator of the effect size are required for the IVhet model, and hence it provides a robust statistical method compared to the REs model.

The above paper also reports that the IVhet model resolves the problems related to underestimation of the statistical error, poor coverage of the confidence interval and increased mean squared error (MSE) usually seen with the REs model. Moreover, unlike the REs, the results of the IVhet model do not depend on the validity of the assumption of normal distribution of both within and between effect sizes. Furthermore, unlike the REs model, the IVhet model does not unrealistically assume that the selected studies included in the meta-analysis are a random sample of all studies in the population.

(d) Quality Effects Model

Doi and Thalib (S.A.R. Doi & Thalib, 2008) introduced a quality-effects (QE) approach that combines evidence from a series of trials comparing two interventions. This approach incorporates the heterogeneity of effects in the analysis of the overall interventional efficacy. However, unlike the random-effects model based on observed between-trial heterogeneity, it suggests adjustment based on measured methodological heterogeneity between studies.

The QE model was updated by (S.A. Doi, Barendregt, Khan, Thalib, & Williams, 2015b) to improve two specific aspects of the model.

First, overdispersion observed with the initial estimator has been corrected using an intra-class correlation based multiplicative scale parameter. Second, the quality scores were originally re-scaled between 0 and 1 for input into the model. Even though they are still rescaled between 0 and 1 but each of these rescaled scores is divided by the maximum value of the rescaled scores within the meta-analysis before it is input into the model. This still keeps the scores in the 0–1 range but allows them to reflect the relative nature of these scores, relative to the best study in the meta-analysis.

Under the QE model the estimator of the common effect size θ is given by

$$\hat{\theta}_{QE} = \sum_{i=1}^{k} w_i^{'}\hat{\theta}_i \text{ , where the modified weight } w_i^{'} = \left(\frac{Q_i}{v_i} + \hat{\tau}_i\right) \Big/ \sum_{i=1}^{k}\left(\frac{Q_i}{v_i} + \hat{\tau}_i\right).$$

For the explanation of the notations, expression of variance of the estimator and other details please refer to the Appendix A of (S. A. Doi et al., 2015b).

3. APPLICATIONS IN HEALTH DATA

In the study of physical activity time of shift workers (SW) and non-shift workers (NSW), (Monnaatsie, Biddle, Khan, & Kolbe-Alexander, 2021) collected and analysed the following data set from 12 primary studies. The sample size, mean time spent of physical activities and its standard deviation for each of the studies for both shift workers and non-shift workers are recorded in the table below.

Table 1
Time Spent in Physical Activities by Shift Workers and Non-Shift Workers

Author	SW N1	SW_ Mean	SW_ StDev	NSW N2	NSW_ Mean	NSW_ StDev
Hulsegge et al., 2017	100	13.9	9	612	15.8	7.3
Lauren et al. 2019	12	12.8	6.7	12	14.3	5.5
Loef et al., 2017	532	18	15.4	5980	15.4	8.8
Loef et al., 2018	401	11.7	6.5	78	12.5	6.2
Loprinzi 2015	219	25.8	3.2	1162	28.5	0.7
Neil-Sztramko et al., 2016	452	10.1	10.1	3061	10.2	6.4
Clark et al., 2017	1223	12.5	11	6612	11.5	9
Peplonska et al., 2014	354	26	11	371	20.9	9.9
Tada et al., 2014	1579	25.2	35.1	1179	24.7	35.7
van de Langenberg et al., 2019	69	5.1	8.1	21	3.6	5.1
Vandelanotte et al., 2015	419	26.8	9.7	775	24.9	9.4
Vlahoylannis et al., 2021	20	15.5	15.7	20	12.4	15.7

The common effect size in the study is the weighted mean difference (WMD) of time spent on physical activities for the shift and non-shift workers. The meta-analyses of the above data set are conducted using three commonly used statistical models – FE, REs and IVhet models and the results are compared.

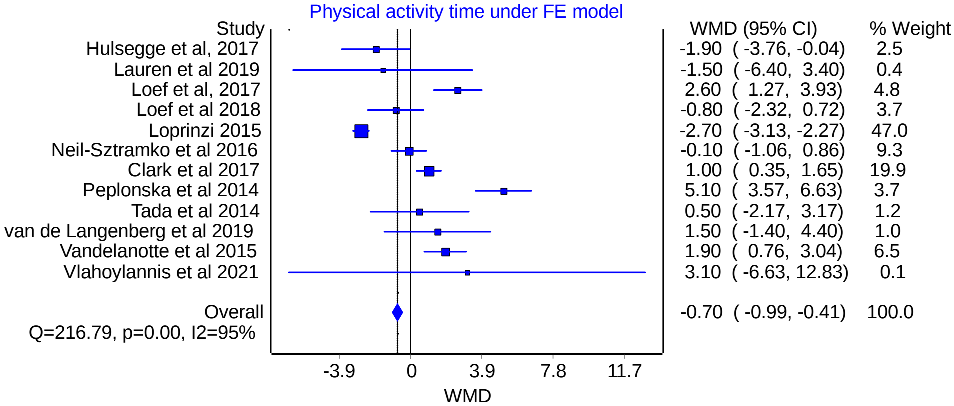

Figure 1: Meta-Analysis of Time Spent on Physical Activities by Shift Workers and Non-Shift Workers using FE Model

The point estimate of the common effect size of WMD under the FE model is -0.70 with the 95% confidence interval (-0.99, -0.41). The effect size is significant with significantly less (by 0.70 units) time spent on physical activities by shift workers compared to the non-shift workers.

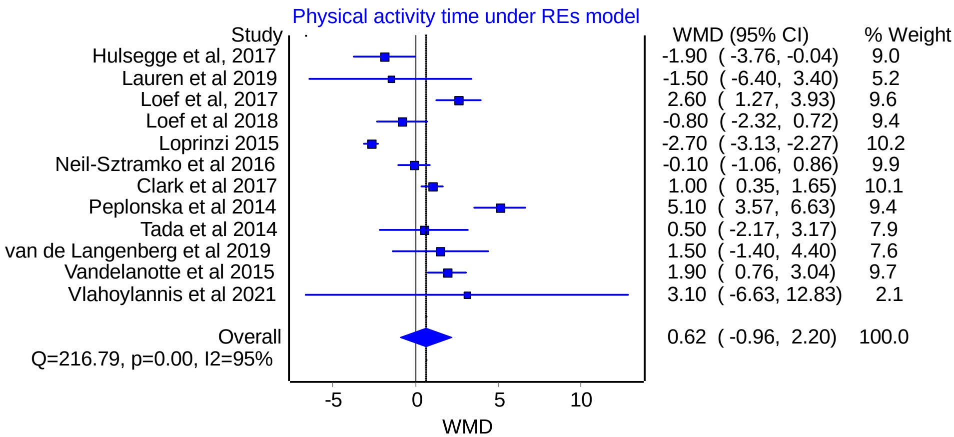

Figure 2: Meta-Analysis of Time Spent on Physical Activities by Shift Workers and Non-Shift Workers using REs Model

The point estimate of the common effect size of WMD under the REs model is 0.62 with the 95% confidence interval (-0.96, 2.20). The effect size is not significant although more (by 0.62 units) time spent on physical activities by non-shift workers compared to the shift workers. It is worth noting that the weight (last column of the forest plot) under the REs model is very different from those in the FE model. Under the REs model the re-distribution of weight resulted in much higher weighs for studies with positive WMD than those under the FE model. This re-distribution of weights has changed the even the sign of the WMD (along with the magnitude).

The weight for the (Loprinzi & Kane, 2015) study under the FE model was 47% and only 10.2% under the REs model. Since the WMD of the study is negative (-2.70), with very high weight under the FE model the estimate of the common effect size has become negative. Also note that the two smallest, second and the last, studies have received much higher weights under the REs model compared to the FE model. This is an example of how the REs model allocates more weights to the smaller studies and less to the larger studies.

Also, the effect size under the RE model is not significant (zero is included in the 95% confidence interval).

It is worth noting that the studies are heterogeneous as evident by the P-value = 0 for the Q statistic and I^2 =95%. So the FE model is in appropriate for the data. But the REs model also has its own problem.

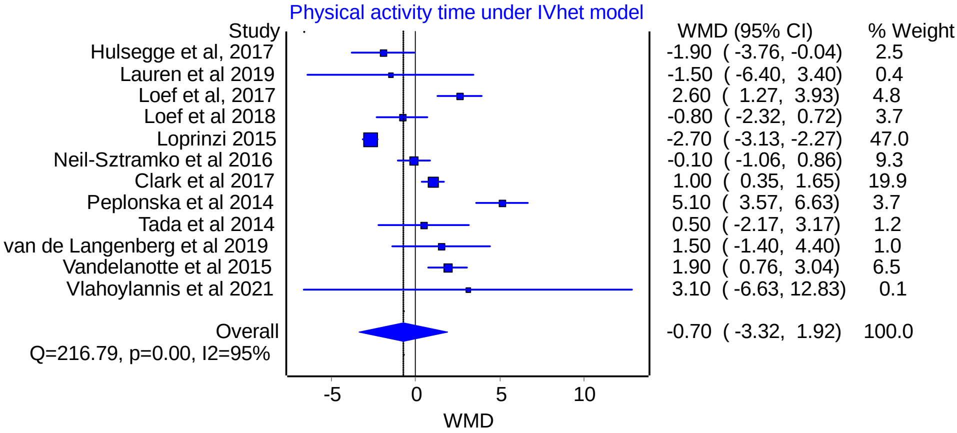

Figure 3: Meta-Analysis of Time Spent on Physical Activities by Shift Workers and Non-shift Workers using IVhet Model

The point estimate of the common effect size of WMD under the IVhet model is -0.70 with the 95% confidence interval (-3.32, 1.92). Although the effect size is not significant it is observed that less (by 0.70 units) time spent on physical activities by shift workers compared to the non-shift workers. It is worth noting that the weights (last column of the forest plot) under the IVhet model are the same as those of the FE model.

Unlike under the FE model, the common effect size under the IVhet model is not significant (zero is included in the 95% confidence interval).

Since the studies are heterogeneous as evident by the P-value = 0 for the Q statistic and I^2 =95%, the FE model is inappropriate and the REs model inappropriately redistributes the weights to the individual studies, the IVhet model provides more reliable analysis of the data.

4. CONCLUSIONS

This paper presents and discusses commonly used statistical models for meta-analysis. It also compares the advantages and disadvantages of the models. Using an example from the health data, the differences under various models are illustrated. The study reveals that the IVhet model provides the best option to meta-analyse the heterogeneous data.

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INVITED PAPER

ON THE SIA-ESTIMATOR OF THE r^{th} MOMENT FOR THE CLASS OF SIA LOG-SYMMETRIC DISTRIBUTIONS

Saleha Naghmi Habibullah
Kinnaird College for Women, Lahore, Pakistan
Email: saleha.habibullah@kinnaird.edu.pk

ABSTRACT

The class of SIA log-symmetric distributions facilitates the development of 'SIA-estimators' which are more efficient than their well-known non-SIA counterparts (i.e. the moment estimators). Whereas the SIA-estimators of the mean and the second moment about zero have already been developed, in this paper, we present the SIA-estimator of the r^{th} moment about the origin. Results of simulation studies have been included in order to present a comparison of the coefficients of variation of the non-SIA estimators of the third and fourth raw moments with the coefficients of variation of the SIA-estimators of these moments. The fact that the superiority of the SIA-estimators of moments about zero over their non-SIA counterparts in term of efficiency leads to *better-fitting models* is demonstrated by referring to an application of the lognormal distribution to a real-world data-set presented in a 2015 paper.

1. INTRODUCTION

Inverted distributions have attracted the interest of researchers, and various inverted distributions have been derived. Distributions that remain unchanged under the reciprocal transformation constitute an appealing *sub-class* of the class of inverted distributions. In the literature, there seem to be only a few distributions that possess the property of invariance under the reciprocal transformation. Gumbel and Keeney (1950), Seshadri (1965) and Saunders (1974) are some of the researchers who investigated the intrinsic properties of such probability models.

After a gap of a little more than three decades, Jones (2008) proposed the term 'log-symmetric' for distributions possessing the property of invariance under the reciprocal transformation whereas Habibullah and Saunders (2011) adopted the term 'self-inverse' for such distributions, and Habibullah (2012) adopted the abbreviation 'SIU' for the term 'Self-Inverse at Unity'. Habibullah and Saunders (2011) showed that the self-inversion at unity property can be regarded as a special case of the property that was later referred to as 'Self- Inverse at A' ('SIA') by Habibullah and Fatima (2015). This property can be stated as follows: The probability distribution of a non-negative continuous random variable X can be regarded as being Self-Inverse at A (SIA) if the distribution of X/A is exactly the same as that of A/X where A can be any positive real number. Habibullah (2017) proposed that the two nomenclatures 'SIA' and 'log-symmetric' be combined in order to obtain the term 'SIA log-symmetric distributions'.

Some of the well-known probability models that belong to the class of SIA log-symmetric distributions are the lognormal, the Burr XII and the Birnbaum Saunders distributions. Other self-inverse probability models include, among others, the Pareto distribution of the second kind, the Beta Prime distribution, the log-logistic, log-Student-t & log-Cauchy distributions as well as the Nakagami-Ratio and the Inverse Gamma-Ratio distributions.

The class of SIA log-symmetric probability models is distinctive in that these models facilitate the creation of estimators that are more efficient than the corresponding well-known moment estimators implying a higher probability of getting better-fitting models for real-world data-sets. Such estimators have been given the name 'SIA-estimators'.

2. A REVIEW OF THE LITERATURE ON SIA-ESTIMATORS

Based on the self-inversion property, a number of 'SIA-estimators' have been proposed during the past few years. Fatima & Habibullah (2013a) proposed self-inversion-based modifications of L-estimators of central tendency in the case of SIA log-symmetric distributions whereas Fatima & Habibullah (2013b) modified L-estimators of dispersion. Habibullah & Fatima (2014a) proposed an SIU-based modification in Kelley's measure of skewness whereas Habibullah & Fatima (2014b) developed a self-inversion-based modification of Crow and Siddiqui's coefficient of kurtosis. Xavier & Habibullah (2015) presented an SIA-estimator of the shape parameter of the log-logistic distribution whereas Ali & Habibullah (2015) developed a new estimator of the scale parameter of the log-Cauchy distribution.

In 2015, Habibullah & Fatima proposed a modification to the formula of the ordinary sample mean for estimating the mean of an SIA log-symmetric distribution. The SIA-estimator of the distribution mean has been used in subsequent research-studies, and has played a non-trivial role in achieving better-fitting models for real-world data-sets. For example, Habibullah (2017a) presented an improved estimator of the location parameter of the lognormal distribution, the objective being to achieve more accurate modeling. Habibullah (2017b) presented the power generalization of SIA log-symmetric distributions.

3. SIA-ESTIMATOR OF r^{th} MOMENT ABOUT THE ORIGIN IN THE CASE OF SIA LOG-SYMMETRIC DISTRIBUTIONS

In this paper, we propose the following SIA-estimator of the r^{th} moment about the origin for the class of SIA Log-Symmetric distributions:

$$\mu'_{r_{SIA}} = \frac{\sum_{i=1}^{n} x_i^r + \hat{A}^{2r} \sum_{j=1}^{n} \left(\frac{1}{x_j^r} \right)}{2n} \tag{3.1}$$

Setting r equal to 1, 2, 3 and 4, we obtain:

$$\mu'_{1_{\mathrm{SIA}}} = \frac{\sum_{i=1}^{n} x_i + \hat{A}^2 \sum_{j=1}^{n} \left(\frac{1}{x_j} \right)}{2n} \tag{3.2a}$$

$$\mu'_{2_{\mathrm{SIA}}} = \frac{\sum_{i=1}^{n} x_i^2 + \hat{A}^4 \sum_{j=1}^{n} \left(\frac{1}{x_j^2} \right)}{2n} \tag{3.2b}$$

$$\mu'_{3_{\mathrm{SIA}}} = \frac{\sum_{i=1}^{n} x_i^3 + \hat{A}^6 \sum_{j=1}^{n} \left(\frac{1}{x_j^3} \right)}{2n}, \tag{3.2c}$$

$$\mu'_{4_{\mathrm{SIA}}} = \frac{\sum_{i=1}^{n} x_i^4 + \hat{A}^8 \sum_{j=1}^{n} \left(\frac{1}{x_j^4} \right)}{2n} \tag{3.2d}$$

It is easy to see that the estimator obtained by putting *r* equal to 1 is the same as the SIA-estimator of the distribution mean given by Habibullah and Fatima (2015). Also, it is worthwhile to note that the four estimators obtained by putting *r* equal to 1, 2, 3 and 4 in the SIA-estimator (3.1) are meant to provide better estimation of the center, spread, skewness and kurtosis of the distribution.

4. ILLUSTRATION OF EFFICIENCY OF SIA-ESTIMATORS OF DISTRIBUTION'S FIRST AND SECOND MOMENTS ABOUT ZERO THROUGH SIMULATION STUDIES

In this section, we first illustrate the superiority of the SIA-estimator of the distribution mean $\mu = E(X)$ over the well-known moment estimator i.e. the ordinary sample mean and then go on to presenting a similar illustration in the case of the SIA estimator of the distribution's second moment about zero i.e. $E(X^2)$. We present the first illustration by referring to a simulation study conducted by Habibullah and Fatima (2015) who drew one thousand samples of size 50 each from the lognormal distribution with μ=2 and σ =1. For each sample, the moment estimator of the mean of the lognormal distribution as well as its SIA-estimator was computed. For each of the two estimators, the 1000 sample results were used to compute the coefficient of range as well as the coefficient of variation of the sampling distribution of the estimator. Focusing on the coefficient of range, in the case of the moment estimator (i.e. the ordinary sample mean), the numerical value of the coefficient turned out to be 0.5480 whereas in the case of the SIA-estimator, the numerical value of the coefficient turned out to be 0.3443. Focusing on the coefficient of variation, in the case of the moment estimator, the numerical value of the coefficient turned out to be 0.1857 whereas in the case of the SIA-estimator, the numerical value of the coefficient turned out to be 0.1049. Thus, it was easy to see that, in this simulation study, the SIA-estimator of the mean of the lognormal distribution was *substantially* more efficient than the ordinary sample mean. The numerical value of the coefficient of variation of the SIA-estimator turned out to be *a little more than half* of the numerical value of the coefficient of variation of the ordinary sample mean. For the table containing the simulation results, see the paper by Habibullah and Fatima (2015).

For the second illustration, we refer to a simulation study conducted by Mushtaq (2022) who has developed a new four-parameter SIA log-symmetric distribution and, along with (i) the median, and (ii) SIA-estimators of the mean and the harmonic mean, has presented the *SIA-estimator of the second moment about zero* for estimating the parameters of the distribution. The newly derived distribution is the result of a linear combination of the SIA log-logistic distribution and the inverse-gamma ratio distribution. The density function of the 'SIA Log-Logistic LC Inverse-Gamma Ratio Distribution' is given by

$$f(x) = mb\mathrm{A}^b x^{b-1}\left(x^b + \mathrm{A}^b\right)^{-2} + (1-m)\frac{\mathrm{A}^c}{\mathrm{B}(c,c)} x^{c-1}\left(x+\mathrm{A}\right)^{-2c}$$

$$= \frac{1}{\mathrm{B}(c,c)}\left[mb\mathrm{B}(c,c)\mathrm{A}^b x^{b-1}\left(x^b + \mathrm{A}^b\right)^{-2} + (1-m)\mathrm{A}^c x^{c-1}\left(x+\mathrm{A}\right)^{-2c} \right],$$

$$0 < x < \infty,\ \mathrm{A} > 0, b > 0, c > 0, 0 < m < 1 \quad (3.3)$$

Table 1
Mean, Standard Deviation and Coefficient of Variation of the Moment Estimator and the SIA-Estimator of the Distribution's Second Moment about Zero

Sample size *n*	**Moment Estimator**	**SIA Estimator**
50	**Mean** of the 75,000 values of the moment estimator of the distribution's 2^{nd} moment about zero = 46424561	**Mean** of the 75,000 values of the SIA-estimator of the distribution's 2^{nd} moment about zero = 24715016
	Standard Deviation of the 75,000 values of the moment estimator of the distribution's 2^{nd} moment about zero = 09	**Standard Deviation** of the 75,000 values of the SIA-estimator of the distribution's 2^{nd} moment about zero = 40
	Coefficient of Variation of the 75,000 values of the moment estimator of the distribution's 2^{nd} moment about zero **= 7427.82%**	**Coefficient of Variation** of the 75,000 values of the SIA-estimator of the distribution's 2^{nd} moment about zero **= 6385.38%**
100	**Mean** of the 75,000 values of the moment estimator of the distribution's 2^{nd} moment about zero = 68566038848	**Mean** of the 75,000 values of the SIA-estimator of the distribution's 2^{nd} moment about zero = 34287599905
	Standard Deviation of the 75,000 values of the moment estimator of the distribution's 2^{nd} moment about zero = 8710664080848	**Standard Deviation** of the 75,000 values of the SIA-estimator of the distribution's 2^{nd} moment about zero = 93553339177.19

Sample size *n*	Moment Estimator	SIA Estimator
	Coefficient of Variation of the 75,000 values of the moment estimator of the distribution's 2^{nd} moment about zero **= 27288.53%**	**Coefficient of Variation** of the 75,000 values of the SIA-estimator of the distribution's 2^{nd} moment about zero **= 27284.89%**
250	**Mean** of the 75,000 values of the moment estimator of the distribution's 2^{nd} moment about zero = 77057207 **Standard Deviation** of the 75,000 values of the moment estimator of the distribution's 2^{nd} moment about zero = 1504577877 **Coefficient of Variation** of the 75,000 values of the moment estimator of the distribution's 2^{nd} moment about zero **= 19525.47%**	**Mean** of the 75,000 values of the SIA-estimator of the distribution's 2^{nd} moment about zero = 55706972 **Standard Deviation** of the 75,000 values of the SIA-estimator of the distribution's 2^{nd} moment about zero = 8423832844 **Coefficient of Variation** of the 75,000 values of the SIA-estimator of the distribution's 2^{nd} moment about zero **= 15121.69%**

Through an R program, Mushtaq (2022) has presented the results of a simulation study based on 75,000 samples of various sizes, each drawn from the above distribution putting A=1, b=1 and c=1. The means, standard deviations, and coefficients of variation of the sampling distributions of the Non-SIA and SIA estimators of the distribution's second moment about zero obtained by Mushtaq (2022) are reproduced in Table 1. The table shows that the coefficient of variation of the SIA estimator of the distribution's second moment about zero is ***smaller*** than the coefficient of variation of the moment estimator for each of the three sample sizes that were chosen for the simulation study.

Through an R program similar to the one used by Mushtaq (2022), we obtain results of a simulation study based on 75,000 samples of various sizes, each drawn from the 'SIA Log-Logistic LC Inverse-Gamma Ratio Distribution' putting A=1, b=1 and c=1. The means, standard deviations, and coefficients of variation of the sampling distributions of the Non-SIA and SIA estimators of the distribution's third moment about zero are presented in Table 2.

Through an R program similar to the one above, we obtain results of a simulation study based on 75,000 samples of various sizes, each drawn from the 'SIA Log-Logistic LC Inverse-Gamma Ratio Distribution' putting A=1, b=1 and c=1. The means, standard deviations, and coefficients of variation of the sampling distributions of the Non-SIA and SIA estimators of the distribution's fourth moment about the origin are presented in Table 3.

Table 2
Mean, Standard Deviation and Coefficient of Variation of the Moment Estimator and the SIA-Estimator of the Distribution's Third Moment about Zero

Sample size *n*	Moment Estimator	SIA Estimator
50	**Mean** of the 75,000 values of the moment estimator of the distribution's 3rd moment about zero = 1.472582 x 10^{16} **Standard Deviation** of the 75,000 values of the moment estimator of the distribution's 3rd moment about zero = 4.010052 x 10^{18} **Coefficient of Variation** of the 75,000 values of the moment estimator of the distribution's 3rd moment about zero **= 27231.43%**	**Mean** of the 75,000 values of the SIA-estimator of the distribution's 3rd moment about zero = 1.158373 x 10^{16} **Standard Deviation** of the 75,000 values of the SIA-estimator of the distribution's 3rd moment about zero = 2.277342 x 10^{18} **Coefficient of Variation** of the 75,000 values of the SIA-estimator of the distribution's 3rd moment about zero **= 19659.84%**
100	**Mean** of the 75,000 values of the moment estimator of the distribution's 3rd moment about zero = 7.366581 x 100^{15} **Standard Deviation** of the 75,000 values of the moment estimator of the distribution's 3rd moment about zero = 2.005026 x 10^{18} **Coefficient of Variation** of the 75,000 values of the moment estimator of the distribution's 3rd moment about zero **= 27217.87%**	**Mean** of the 75,000 values of the SIA-estimator of the distribution's 3rd moment about zero = 5.794318 x 10^{15} **Standard Deviation** of the 75,000 values of the SIA-estimator of the distribution's 3rd moment about zero = 1.138671 x 10^{18} **Coefficient of Variation** of the 75,000 values of the SIA-estimator of the distribution's 3rd moment about zero **= 19651.51%**
150	**Mean** of the 75,000 values of the moment estimator of the distribution's 3rd moment about zero = 4.912387 x 10^{15} **Standard Deviation** of the 75,000 values of the moment estimator of the distribution's 3rd moment about zero = 1.336684 x 10^{18} **Coefficient of Variation** of the 75,000 values of the moment estimator of the distribution's 3rd moment about zero **=27210.48%**	**Mean** of the 75,000 values of the SIA-estimator of the distribution's 3rd moment about zero = 3.865515 x 10^{15} **Standard Deviation** of the 75,000 values of the SIA-estimator of the distribution's 3rd moment about zero = 7.591141 x 10^{17} **Coefficient of Variation** of the 75,000 values of the SIA-estimator of the distribution's 3rd moment about zero **=19638.11%**

Table 3
Mean, Standard Deviation and Coefficient of Variation of the Moment Estimator and the SIA-Estimator of the Distribution's Fourth Moment about Zero

Sample size *n*	**Moment Estimator**	**SIA Estimator**
50	**Mean** of the 75,000 values of the moment estimator of the distribution's 4^{th} moment about zero = 5.570788 x 10^{23} **Standard Deviation** of the 75,000 values of the moment estimator of the distribution's 4^{th} moment about zero = 1.524141 x 10^{26} **Coefficient of Variation** of the 75,000 values of the moment estimator of the distribution's 4^{th} moment about zero = **27359.53%**	**Mean** of the 75,000 values of the SIA-estimator of the distribution's 4^{th} moment about zero = 4.036622 x 10^{23} **Standard Deviation** of the 75,000 values of the SIA-estimator of the distribution's 4^{th} moment about zero = 8.316424 x 10^{25} **Coefficient of Variation** of the 75,000 values of the SIA-estimator of the distribution's 4^{th} moment about zero = **20602.43%**
100	**Mean** of the 75,000 values of the moment estimator of the distribution's 4^{th} moment about zero = 2.785459 x 10^{23} **Standard Deviation** of the 75,000 values of the moment estimator of the distribution's 4^{th} moment about zero = 7.620706 x 10^{25} **Coefficient of Variation** of the 75,000 values of the moment estimator of the distribution's 4^{th} moment about zero = **27358.89%**	**Mean** of the 75,000 values of the SIA-estimator of the distribution's 4^{th} moment about zero = 2.018351 x 10^{23} **Standard Deviation** of the 75,000 values of the SIA-estimator of the distribution's 4^{th} moment about zero = 4.158212 x 10^{25} **Coefficient of Variation** of the 75,000 values of the SIA-estimator of the distribution's 4^{th} moment about zero = **20602.02%**
150	**Mean** of the 75,000 values of the moment estimator of the distribution's 4^{th} moment about zero = 1.856993 x 10^{23} **Standard Deviation** of the 75,000 values of the moment estimator of the distribution's 4^{th} moment about zero = 5.08047 x 10^{25} **Coefficient of Variation** of the 75,000 values of the moment estimator of the distribution's 4^{th} moment about zero = **27358.58%**	**Mean** of the 75,000 values of the SIA-estimator of the distribution's 4^{th} moment about zero = 1.345624 x 10^{23} **Standard Deviation** of the 75,000 values of the SIA-estimator of the distribution's 4^{th} moment about zero = 2.772141 x 10^{25} **Coefficient of Variation** of the 75,000 values of the SIA-estimator of the distribution's 4^{th} moment about zero = **20601.16%**

5. USEFULNESS OF THE SIA-ESTIMATOR OF THE r^{th} MOMENT IN ACHIEVING A BETTER-FITTING MODEL

The fact that, more often than not, SIA-estimators of moments about the origin turn out to be more efficient than their non-SIA counterparts points to their usefulness in achieving better-fitting models. This has been demonstrated by Habibullah and Fatima (2015) as well as by Habibullah and Xavier (2018) for the case $r = 1$. Habibullah and Fatima (2015) applied the SIA-estimator given in (3.2a) to a set of data given by Fan and Fan (2015) pertaining to a contractor's equipment fleet that worked on 3-shift schedule around the clock, and consisted of complete working records of downtime, uptime, failure events, and repair details on each unit. Habibullah and Fatima (2015) picked up 29 values of time to repair (TTR) data of one piece of construction equipment out of the 30 values that were given by Fan and Fan (2015). The one that was left out was 0.00. (This value was discarded in order to be able to compute the SIA-estimator of the mean, the formula of which includes the multiplicative inverse of each observation.)

The 29 values of the TTR data that were picked up are as follows:

3.20, 11.58, 38.37, 1.83, 6.52, 27.43, 15.93, 1.02, 0.50, 7.25, 1.17, 27.40, 92.90, 62.77, 0.33, 0.33, 0.95, 8.58, 1.00, 0.50, 13.83, 1.72, 10.25, 5.25, 40.02, 31.93, 88.27, 0.50, 4.35.

The histogram of this data-set exhibited a highly positively skewed shape. As such, Habibullah and Fatima (2015) chose to fit the lognormal distribution to this data-set. Using the median of the data which turned out to be 6.52 and the ordinary sample mean which turned out to be 17.4372, the value of the K-S statistic D turned out to be 0.200 which was smaller than the critical value 0.246 at 5% level of significance which meant that the lognormal distribution with these values of the parameters turned out to be a good fit. On the other hand, taking median equal to 6.52, the SIA-estimator of the distribution's mean came out to be 22.446 and hence, the value of the K-S statistic D came out to be 0.173 which was ***less than*** 0.200, the value that was obtained using the simple arithmetic mean. As such, Habibullah and Fatima (2015) arrived at the conclusion that the lognormal distribution attained through the use of the SIA-estimator of the mean fitted the TTR data ***better*** than the lognormal distribution attained through the use of the ordinary sample mean.

Similar results can be expected when modeling real-world data through the utilization of SIA log-symmetric distributions in conjunction with SIA-estimators of the second, third, fourth and higher moments about the origin, as need be.

6. CONCLUDING REMARKS

SIA log-symmetric distributions constitute an interesting class of distributions of non-negative continuous random variables as these distributions facilitate the construction of 'SIA-estimators' which are more efficient than the corresponding estimators obtained by the method of moments. The SIA-estimator of the distribution mean was developed in 2015 and the SIA-estimator of the distribution's second moment about zero has been developed very recently. In this paper, we have presented the SIA-estimator of the r^{th} raw moment which is, evidently, a generalization of the previously developed two estimators.

Results of simulation studies that have been included in the paper clearly indicate the superiority of the SIA-estimators over their non-SIA counterparts in term of efficiency. The fact that this desirable property leads to *better-fitting models* is demonstrated by referring to an application of the lognormal distribution to a real-world dataset presented in a 2015 paper.

Formulation of the SIA-estimator of the r^{th} moment about the origin is *significant* as it will lead to SIA-estimators of the second, third and fourth moments about the mean as well as of the moment-ratios β_1 and β_2 which, in turn, is likely to lead to better-fitting models for a large number of data-sets.

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HEALTHCARE IN PAKISTAN

Imran Anwar Ujan[1], **Asadullah Shah**[2] and **Arifa Bhutto**[1]
[1] Faculty of Engineering & Technology, University of Sindh
Jamshoro, Pakistan. Email: iujaniium@gmail.com
[2] Kulliyyah of Information and Communication Technology (KICT)
International Islamic University, Malaysia.

ABSTRACT

Pakistan's health system has evolved over time, with roots in mediaeval medicine, conventional health care, health for population, primary health care, and health system supporting for improved health outcomes. The major goals of the health-care system are improved health, fair risk and financial distribution, and receptiveness to the population's non-medical demands. Declining health-care spending, an updated private health-care sector, and a thriving pharmaceutical production, the government can only decrease calamitous health-care costs for the poor and underprivileged by ensuring that the public health system is efficient, effective, accessible, and responsive. Inter-sectoral association, community contribution, social fortification, equitable resource delivery, people-centered health policy, health workforce advancement, evidence-based health information system, and essential medicine quality confidence will all help to strengthen Pakistan's health system.

INTRODUCTION

The goal of HIS applications is to improve the management of health protection because its utilization will address the limitations in providing health care namely, (a) a more expensive health care, (b) increase of malpractice incidents, and (c) the setbacks encountered in the knowledge diffusion process. Therefore, studies in the future should focus on these issues (Cheng et al., 2009).

Pakistan estimated land area is 881,913 square kilometers. Pakistan is the Muslin Country who have third most populous in the world. Total population of Pakistan is 208.57 million.

The federal budget for 2021-22 increased by 11% over the previous year. In nominal terms, the health budget has increased from around Rs. 25.5 billion to Rs. 28.3 billion, though it still accounts for only 0.4 percent of the total budget. The federal development budget (PSDP) has a "special emphasis on strengthening the health sector" and has increased by 49.6 percent, from Rs. 14.5 billion in 2020-21 to Rs. 21.7 billion, including Rs. 5.6 billion for the Sehat Sahulat Program--me. Impact mitigation of Covid-19 is one of the government's priorities for the current fiscal year. Only Rs. 100 billion is set aside for Covid-19-related expenses, which is nearly four times the budget for routine healthcare. The allocation of the national budget for hospitals and healthcare services falls under this category.

Reason to Conduct Research on Pakistan

Although the allocated budget for investment in the health sector was increased in Pakistan, the rate of utilization and acceptance of IT application by physicians proved to be less than what was expected. A review of literature revealed that the setting of previous studies conducted on technology acceptance and the attitude of users towards it were in developed countries. These countries differ, specifically, in culture to Pakistan. Conducting this study in Pakistan is, therefore, an opportunity to test the reliability and validity of prevailing theories regarding (a) technology acceptance, and (b) attitude towards technology from the perspective of developing countries.

The healthcare sector is critical to the development of a prosperous, stable economy and society. However, the health sector in Pakistan is experiencing from neglect. It is a low-performing industry. Only 27% of the populace had access to full health services and services, even though 73% paid their own medical bills. Due to the 46% reduction in the allocation of funds for health in 2010-2011, Pakistan has the lowest budget for the health sector in the region. The population of Pakistan, however, was observed to be rapidly increasing. In 2000, the reported population of Pakistan was at 142 million. However, in 2011, Pakistan already had a reported population of 187 million with the ratio of one physician is to 1,326 people. In addition to the lack of funding and growing population, other limitations prevalent in the health care of Pakistan are (a) tradition modes of practice, (b) rising frequency of medical errors, (c) delay in knowledge diffusion process, and (d) the interests of private individuals and institutions (Sultan et al., 2014) The use of ICT application in conjunction with evidence-based medical practice is the best way to address the significant challenges that Pakistan is experiencing in health care, namely, (a) compounding cost for health care, (b) rising rate of medical errors, and (c) pervading delay in the knowledge diffusion process (Wears and Berg, 2005).

Pakistan offers both public and private health care. The private sector handles 80 percent of outpatient visits. Welfare and community hospitals also play an important role in providing low-cost health care. The current situation can be developed by implementing information technology in the health sector and training physicians to maximise the benefits of these resources for evidence-based medicine. (Shaikh and Hatcher, 2005) In 2007, 133933 doctors were registered with Pakistan Medical and Dental Council (PMDC) and by 2010; the country had only 54 qualified health informatician. In individual research article it is stated: “Pakistan does not have an institutional mechanism for healthcare service quality control, hospital accreditation or provider credentialing except for the PMDC, which serves the role of registration only.”

In Pakistan, there are 175,223 licenced doctors. From 2011-12 to 2014-15, there was a 13% growth in the overall number of registered doctors, according to a simple analysis. In the same way, there are currently 15,106 registered dentists in the country, up from 11,649 in 2011-12. As a result, the overall number of dentists has increased by 23% over the last several years.

Health Manpower	2011-12	2012-13	2013-14	2014-15
Registered Doctors	152368	160880	167759	175223
Registered Dentists	11649	12692	13716	15106
Registered Nurses	77683	82119	86183	90276

According to Government of Pakistan Economic Survey 2020-21.

Public Hospitals

In the year 2020, the country will have 1,282 public hospitals. That represents three more hospitals were included to the healthcare last year, bring in the total number of government-run hospitals to 1,279 in 2019.

Healthcare Workers

In 2020, 245,987 were licensed practitioners, 27,360 licensed dentists, and 116,659 licensed nurses. When there were 220,829 registered doctors, including at private and public institutions, 22,595 dentists, and 108,474 nurses, in the 2019 the numbers have increased.

Life Expectancy

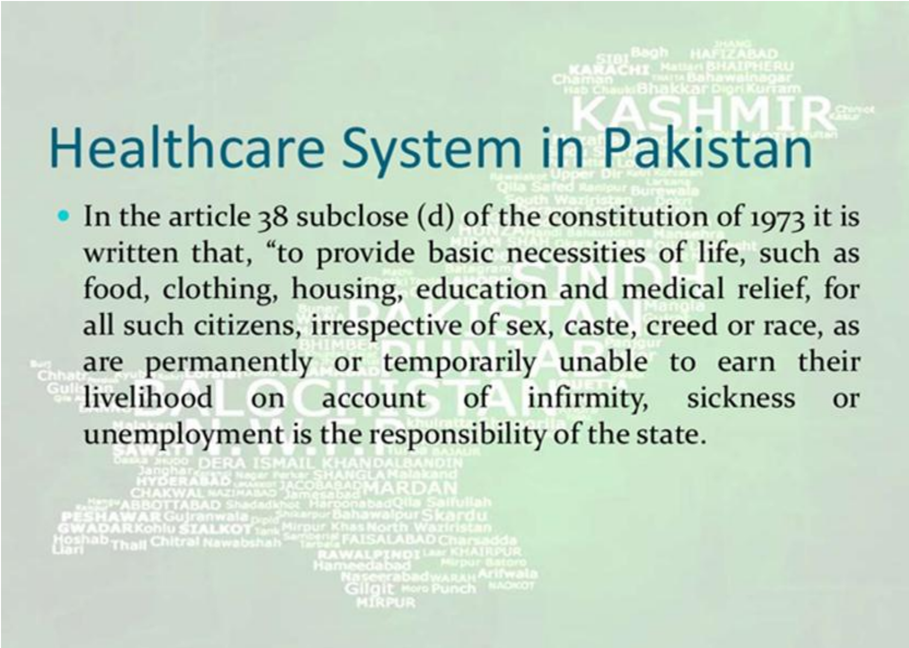
Life expectancy in the country has risen from 66.9 years in 2017 to 67.3 years in 2019.

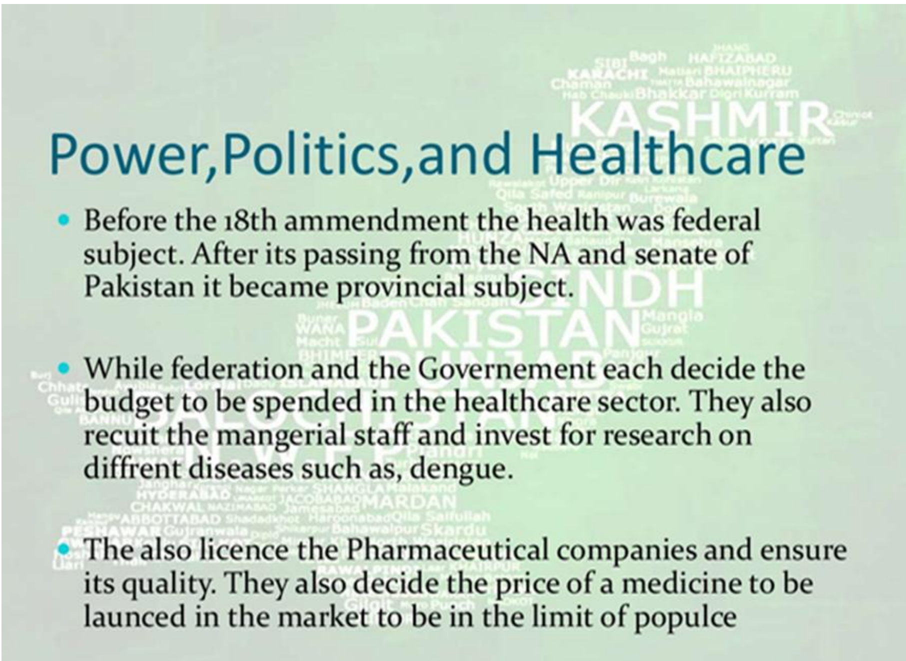
Population Growth

The population increase rate has slowed, falling from 2% in 2018 to 1.9 percent this year. Country birth control device popularity rate remained unchanged in 2019 at 34%.

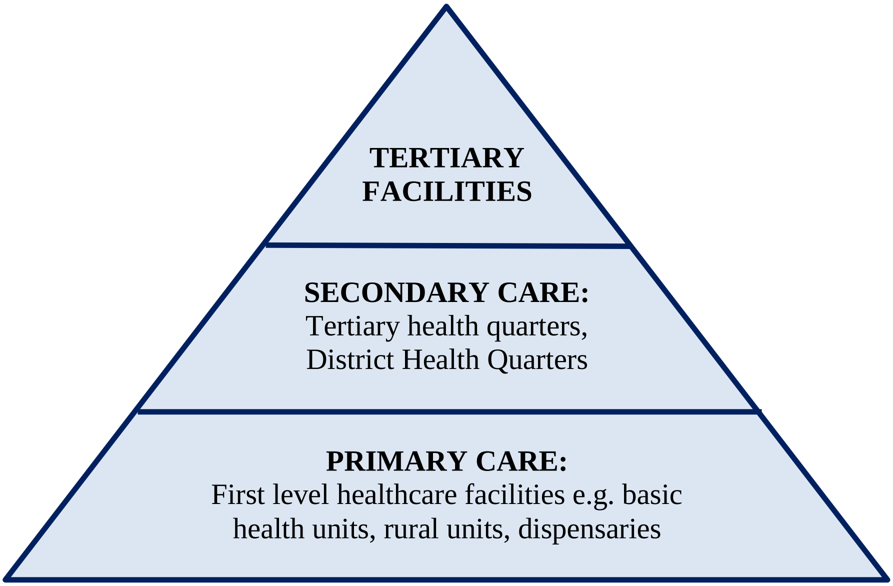
Health Expenditure

Pakistan spent 1.2 percent of its GDP on health in 2020, far less than the World Health Organization's recommendation of 5%.


Healthcare System in Pakistan
In the article 38 subclose (d) of the constitution of 1973 it is written that, “to provide basic necessities of life, such as food, clothing, housing, education and medical relief, for all such citizens, irrespective of sex, caste, creed or race, as are permanently or temporarily unable to earn their livelihood on account of infirmity, sickness or unemployment is the responsibility of the state.


Power,Politics,and Healthcare
Before the 18th ammendment the health was federal subject. After its passing from the NA and senate of Pakistan it became provincial subject.
While federation and the Governement each decide the budget to be spended in the healthcare sector. They also recuit the mangerial staff and invest for research on diffrent diseases such as, dengue.
The also licence the Pharmaceutical companies and ensure its quality. They also decide the price of a medicine to be launced in the market to be in the limit of populce

Healthcare System: Organization



CONCLUSION

The critical to establish robust inter-sector activities, rules and standards for healthcare transfer, quality assurance in the pharmaceutical industry, and collaboration with the private sector health care in Pakistan to achieve overall improvements in health, impartiality in risk delivery and funding, and receptiveness to the non-medical requirements of the clientele by the healthcare system. A strong stewardship function is required to give guidance and effective oversight. The government's stewardship role in introducing good governance and promoting better responsiveness in Pakistan's health system might be strengthened through public-private partnerships.

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FORMATION OF INTELLECTUAL CAPITAL THROUGH SOCIAL CAPITAL

Sahar Latif Rana[1§], **Nadia Hanif**[2] and **Abida Hafeez**[3]
Division of Management and Administrative Science
University of Education, Lahore, Pakistan
Email: [1]sahar.latif@ue.edu.pk
[2]nadia.hanif@ue.edu.pk
[3]abida.hafeez@ue.edu.pk
[§]Corresponding author

ABSTRACT

Theory of firm has played significant role in the determination of causes and consequences of market failure. Researchers instead of focusing on identifying these consequences of market failure, pay importance to gaining competitive advantage and enhancing organizational performance. Typically, researchers are of the view that organizations can attain such advantage by using existing capabilities and sharing knowledge. This article seeks to contribute to this body of work by developing following arguments. Does social capital leads toward the formation of intellectual capital? A model is presented that include these arguments which are established on the basis of hypothesized relationships between different dimensions of social capital and process necessary for the creation of intellectual capital. An extensive literature review was done for data collection. The study reveals that by combining and sharing an existing array of knowledge, firms or organizations will become more profitable and competitive. This article has focused only on process of creation of intellectual capital, not on its content, hence it is considered one of its limitations. The future research is directed to empirically validate the proposed model, in order to ensure generalizability.

KEYWORDS

Social capital, intellectual capital, trust, knowledge sharing, goal congruence.

INTRODUCTION

Sayyadi (2021), stated “that a firm be understood as a social community specializing in the speed and efficiency in the creation and transfer of knowledge”. Rui and Bruyaka, (2021) observed that this new concept of creation and transfer of knowledge on theory of firm being formalized by several researchers. The current study is grounded on the basis of social capital which is related to establishments of relationships (Baker, 1990). According to Putnam (1995), social capital is not a one-dimensional notion, whereas it is a combination of different aspects of social capital. In this study, (1) social capital is elaborated in terms of its different dimensions and (2) Determine how these dimensions help in the formation of intellectual capital.

Social Capital

Mngumi (2021), opined that origin of the term "social capital' is in community studies, which actually highlights functioning and survival of city neighborhoods where personal relationships are established on the basis of cooperation, trust and collective actions in such communities. Coleman (1988) observed that the concept of social capital is basically a social phenomenon, where researchers have considered this phenomenon as an influence, not as a development of human capital. Song and Chen (2021), defined social capital as: "the collectivity-owned capital, a 'credential' which entitles them to credit, in the various senses of the word". Furthermore, he was of an opinion that acquaintances, family members, class fellows are responsible for the development of feelings of respect, friendship and gratitude. Although, these researchers consider social capital as an establishment of relationships but they have different opinions regarding its definitions. Putnam (1995), stated that there is a need to study dimensions of social capital. In order to explore these dimension, this study has classified these dimensions into three categories, relational, structural and cognitive dimensions. In spite of categorizing these dimensions into three categories, yet many of their features are highly interrelated. Wang and Le (2021) observed a difference between structural and relational dimension of social capital. Barrutia and Echebarria (2021) stated that structural dimension is regarded as an impersonal formation of linkages between people or units. Structural dimension for this article considers establishment of connections through different patterns- that is to whom you reach and how to reach them Scott (1991), determines the important aspects of this dimension, particularly network configuration, and network ties.

Granovetters (1992), stated that relational dimension means, establishment of relationships through number of interactions. It basically include particular relationships such as friendship and respect. This study considers relational dimension as relationships affecting behavior Lindenberg (1996), whereas structural dimensions refer to "actor bonds". This cluster comprise of following key facets, obligations and expectation (Burt, 1992; Coleman 1990), Trust (Putnam, 1993), norms (Coleman, 1990). Cicourel (1973) stated Cognitive dimension of social capital "refers to those resources providing shared representation, interpretations and systems of meaning among parties". Burt (1992), stated that social capital is all about establishing relations between persons and among persons, in social capital individual will not enjoy ownership rights.

Intellectual Capital

Traditional viewpoint about physical and human activities in achieving organizational goals have been refuted by many economists. They now state that knowledge also play a major role in achieving organizational capital. Watson and Mathew (2021), opined that, "capital consists in a great part of knowledge and organizations… knowledge is our most powerful engine of production." Quinn (1992), described the similar viewpoint, "with rare expectations, the economic and producing power of the firm lies more in its service and intellectual capabilities than its hard assets- equipment, plant and land. Virtually, all private and public enterprises – including most successful corporations- are becoming dominantly repositories and coordinators of intellect." for the sake of this study, intellectual capital is "referred as knowledge and knowing capability of a social collectivity such as professional practice, intellectual community, and an organization".

This terminology is opted as it is parallel to human capital which is a mixture of skills, knowledge, capabilities, used by human to perform certain activities in an entirely new manners. (Coleman, 1988). As the term intellectual capital is used in this article to represent knowledge and knowing, so it is necessary to distinguish between different types of knowledge and particularly to answer a question that in what form social or collective knowledge exist?

Types of Knowledge

Polanyi (1967), stated two types of knowledge: explicit and tacit, the distinction is made between these types on the basis of "knowing how" and "knowing what'. Further he stated that tacit knowledge is not communicated, it remain hidden in human mind. He exemplified this knowledge as the knowledge gained over a period of time. Knowledge not only exist at individual level, rather it can also be accumulated at collective level. For example, organizations are a great source of individual as well as collective knowledge. Dong and Crossan (2021), observed that "all organizational learning take place inside human heads; an organizations learn only in two ways (a) by the learning of its members or (b) by ingesting new members who have knowledge the organization didn't previously have". Spender (1996), who classified knowledge in four different dimensions and considered them as dimensions of organizational social capital, these are individual explicit knowledge- "conscious knowledge" available in the form of facts, or frameworks, or retrieved form memory. The other elements are individual tacit knowledge referred by Spender (1996) as "automatic knowledge', social explicit knowledge (objectified knowledge) and social tacit knowledge (collective knowledge). This study will be limited to social knowledge only, as the main focus of this article is to determine how social capital aids in the formation of new intellectual capital.

Intellectual Capital Formation

The question how new knowledge is created has been answered by Schumpeter (1934), Moran and Ghoshal (1996) in following manner, that two important processes combination and exchange helps in the creation or amalgamation of existing or new knowledge. Creation of new knowledge can be incremental or radical. Incremental change is utilizing existing knowledge to create new knowledge (Kuhn, 1970). On the other hand radical change means innovation, in Kuhn (1970) terms it means revolution. Hence combination may be radical in a sense that combining existing elements unconnected to create new knowledge or incremental, by developing novel ways of combining elements previously associated. Like combination, exchange is also an important element required for the formation of intellectual capital. Combination and exchange are two building blocks of creation of intellectual capital. Zucker (1996) stated that co activity and social interaction helps in the formation of intellectual capital. Peng (1996) had highlighted the importance of collaboration in the development of knowledge in the field of bio technology. Penrose (1959), theory stated that firm can be viewed as "a collection of individuals who have had experience in working together, for only n this way can 'teamwork' be developed."

Ghoshal and Moran (1996), stated that three conditions need to be fulfilled for combination and exchange. The first condition is existing of an opportunity to make combination or exchange. Crane (1972), stated that latest developments in the field of technology like I phones, tabs and internet have provided platform for the combination and exchange of intellectual capital. The other condition is that the combining and exchanging knowledge must add value. In other words, people may think that exchange and combination will prove worthwhile. The third condition is that motivational factor must exist for combination and exchange, the parties must consider that their involvement will worth their while. Cohen (1990) stated that their exist another condition i.e. combination capability. Even if there is an opportunity and parties may feel that their combination and exchange will prove worthwhile and also worth their while, but they have no capability to combine and exchange their knowledge then their all efforts will be in vain.

Exchange and Combination: Precursor to Social Capital

Social capital is based on relationships and exchange is the only key to the establishment of relationships (Bourdieu, 1986). Exchange is an antecedent to social combination of resources in social system. Several facets of social capital have direct influence on individual's ability to combine knowledge for the sake of creation of intellectual capital. As discussed earlier, social capital comprise of three dimensions, structural, relational and cognitive. Now the next step is to determine, how the facets of these dimensions leads to the formation of intellectual capital. The main argument was that social capital helps to create intellectual capital by combination and exchange of knowledge. To explore this argument impact of each dimension of social capital was observed independently of other dimensions

STRUCTURAL DIMENSION AND FORMATION OF IC

The main argument in this section is to determine how structural dimension of social capital through exchange and combination helps in the creation of new intellectual capital. as the structural dimension is further characterized in to two domains network ties and network configuration. Network ties means the ways which provide access to resources. Coleman (1988), opines that no doubt, information is a rudiment of any action to be performed but it is hardly to gather. For combining all information at one place, network ties or social relations are fruitful. Social relations, although developed for different purposes, laid down information channels hence reducing time and money needed to collect information. Burt (1992), stated timing, referrals and access are three forms which helps to attain benefits from information. By term "access" it means having valuable source of information and determining identifying who can use it. Thus network ties have influence on two conditions of combination and exchange. These are access to parties and anticipation of value. Burt (1992), further stated that as network ties enable resources to access information whereas the overall configuration of these ties develop an important facet of social capital that have an impact in the development of intellectual capital.

COGNITIVE DIMENSION AND FORMATION OF IC

As defined earlier, that "intellectual capital is knowledge and knowing capability of a social collectivity." According to some researchers, knowledge sharing require some sharing of context between parties (Campbell, 1969). According to cognitive dimension, this sharing can occur through the existence of shared codes and languages. There are number of ways by which language have impact on conditions of combination and exchange. First, language is directly related with the sharing of knowledge. To the extent people share common language, the more they have an access to parties and gain information. Burger and Luckman (1996), opined that shared languages enable parties to evaluate the valuable resources of information for key benefits of combination and exchange. Nonaka (1995) studied that advancement in knowledge takes place through development of new concepts and narrative forms. Hence shared language enhances combination capability.

RELATIONAL DIMENSION OF SOCIAL CAPITAL AND CREATION OF IC

According to prior researches, relational dimensions of social capital has an influence on three conditions of combination and exchange. These are access to parties, motivation of parties to combine and exchange their knowledge and anticipation of value. The term "Trust" in relational dimension of social capital is defined by Misztal (1996) as "results of somebody's intended action will be appropriate from our point of view." Putnam (1993) stated that higher the level of trust, higher will be established social relations and thus provides access to parties. Mishira (1996) argued that trust is a construct which is multidimensional in nature and it shows a readiness to be susceptible to other parties. Misztal (1996), observed that "trust, by keeping our mind open to all evidence, secures communication and dialogue." And thus provide both access to parties t combine and exchange and anticipation of value.

According to Coleman (1990), "a norm exist when the socially defined right to control an action is held not by the actor but by the other. Thus, it represents a degree of consensus in the social system. He suggested where a norm exist and is effective, it constitutes a powerful though sometimes fragile form of social capital." norms of an organization laid basis for the creation of intellectual capital. Kramer and Goldman (1995), stated that norms have a significant impact on exchange processes. Providing access to parties to combine knowledge ND ensuring motivation to indulge in such exchange. Coleman (1990), defined obligations, "a commitment or a duty to undertake some activity in the future." In the context of this article, obligations provide both access to parties to combine knowledge and motivate them to combine and exchange knowledge. Thus, far it can be concluded that theory of social capital laid down solid grounds for understanding formation of intellectual capital.

DISCUSSION AND PRACTICAL IMPLICATIONS

Theory of firm believes that firms instead of establishing social relations, merely focus on gaining competitive advantage and high financial growth. This belief has been refuted by theory of social capital which describes that establishing social relations will help firm to grow both financially as well as competitively. This article is based on arguments that whether social capital facilitates the creation of intellectual capital? To explore this argument, resource based theoretical framework by Moran and Ghoshal (1996) was used. It was argued that new intellectual capital is owed to the combination and exchange of existing intellectual resources, which may either be tacit or explicit. These dimensions of knowledge are grouped into matrix by Spender (1996), as "individual tacit knowledge, individual explicit knowledge, social tacit knowledge and social explicit knowledge". For this article, only social tacit and explicit knowledge is considered, hence it is one of limitation of this paper. Further, it is stated that for the creation of new intellectual capital, a process of combination and exchange is required which is based on four conditions, access to parties, motivation, and value anticipation and combination capability. It is evident from extensive literature that those firms who invest more on establishment of social capital are more successful. Extensive literature review, visualized that social capital has categorized into three dimension which collectively, target combination and exchange process and ultimately lead to the establishment t of new IC.

LIMITATIONS & FUTURE DIRECTIONS

This study is also not free of limitations. First, regarding social capital the main focus was on the creation of intellectual capital, negative impacts of social capital have been ignored. Regarding intellectual capital, focus was on its creation, not on its contents and types. Instead of assessing three dimensions of social capital collectively, they are assessed separately. The study can be extended in various ways. Future researchers can study the inter relationships among three dimensions of social capital. Beside creation of intellectual capital, the other important aspects like diffusion and exploitation are needed to be researched. The study of content of intellectual capital that is "know how" must be directed for future research. The model outlined here will prove to be beneficial for those seeking to build or extend their network of connections, and thus increasing their stock of social capital. as Bourdieu observes, "the existence of connections is not a natural given, or even a social given…it is the product of an endless effort at institution." The future researchers must empirically verify the contents of this study, in order to ensure its generalizability.

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GENERALIZATION OF THE DIFFERENTIAL TRANSFORM METHOD FOR SOLVING HIGHER ORDER BOUNDARY VALUE PROBLEMS AND ITS APPLICATIONS

Ali Hassan[1], **Muhammad Usman**[2], **Haseeb Mushtaq**[3] and **Hina Saleem**[4]
Department of Mathematics and Statistics, University of Agriculture
Faisalabad, Pakistan
Email: [1]iamalibajwa@gmail.com
[2]usmann75@gmail.com
[3]hinasaleemuaf@gmail.com
[4]nomanjutt136@gmail.com

ABSTRACT

This paper considers higher-order boundary-value problems as well as their approximate solutions are obtained through a new-generalization of differential transformation method (DTM), that is α-Parameterized Differential-Transform Method (α-PDTM). Determining the best DTM modification give rapid and accurate results. Using the proposed methodology to higher order boundary value problems at a various value of parameter α. The analysis is conducted by testing the method both on homogenous and non-homogenous problems. Then obtained results are compared with exact and differential-transformation method (DTM) solutions. In addition, graphical illustration of results is also provided. The suggested method produced remarkably precise and consistent results.

KEYWORDS

Differential-Transform Method, Homogenous and Non-Homogenous Equations, Boundary-Value Problems, Series Solutions.

INTRODUCTION

Many mathematical methods for computing linear or nonlinear differential equations, which represent many significant phenomena and applications in science, have been invented by researchers. The term numerical methods refer to ways for approximating mathematical operations. Because the techniques cannot be solved analytically, approximations are required. Initial and Boundary-value problems are common in engineering and scientific applications, and they are especially common when solving differential equations. Mathematical approaches are essential instruments in the study of mathematics. Numerous numerical methods considerable effective like reduced-differential transformation method [1], differential-transformation method (DTM), variational-iteration method (VIM), galerkin method, homotopy analysis method (HAM), shooting method, adomian-decomposition method, rationalized Haar functions method, HPM and finite-difference method are used for this purpose (see more [4-13]).

Analytical solutions make it possible for researchers to readily explore the effect of many factors on which the function or under consideration. Among the approaches discussed above, the Differential-Transformation method (DTM) stands out for its ease of use and flexibility in solving differential equations. Unlike other common approximative approaches, that approach does not reliant on disorder parameter. The differential-transformation method (DTM) firstly proposed in engineering industry just as tool for solving initial and boundary value problem for calculating approximate solutions to electrical circuits [14]. The significant benefit of the suggested method is that it only requires a few simple steps.

That technique decreases amount of computing material required and can use to a wide range of nonlinear-physical problems. It approximates precise solutions that are sufficiently differentiable using the form of polynomials. DTM generates approximations using an iterative technique based on the Tayler-series expansion. To get the high-order Taylor series (TS), differential transformation approach gives iterative procedure. As a result, it can be used in high-order cases. This approach is extremely effective and significant for solving a broad range differential-equations like Two-point boundary value problem [3], fractional-differential equation [2], differential algebraic equations the Schrodinger equations and much more.

This article, we enlarge a new-generalization of DTM as α-parameterized differential transformation method (α-PDTM). Higher-order boundary value problems are addressed using (α-PDTM) for various parameter values and the methods accuracy and efficiency are demonstrated and compared numerically and graphically with other methods.

Basic Concepts Generalization of DTM

Consider $-\infty < a < b < \infty$, where $g:[a,b] \to \mathrm{R}$ be an analytic function, $U_r(g,t_o)$ represented the differential transform of the $t_o \in [a,b]$ is the *rth*-derivative of function $g(t)$ is defined as [3]

$$U_r(g,t_o) = \frac{1}{r!}\frac{d^r g(t)}{dt^r}\bigg|_{t=t_o}$$

here $g(t)$ is term as original function and $U_r(g,t_o)$ is a transformed function. Differential inverse transformed of $U_r(g,t_o)$ described in following ways:

$$g(t) = \sum_{r=0}^{\infty} U_r(g,t_o)(t-t_o)^r, r = 0,1,2,...$$

and let, for $\alpha \in [0,1]$

$$D(g,\alpha;k) = \alpha U_k(g,a) + (1-\alpha)U_k \;\; \forall, r = 0,1,2,...$$

Definition 1: The sequence

$D_\alpha(g) = (D(g,\alpha;0), D(g,\alpha;1),...)$ is called α -parametrized differential transform (α -PDT) of g.

Let $\mathrm{H}=(h_0,h_1,h_2,...)$ be a real sequence. Then,

$$\chi_\alpha(\mathrm{H},t)=\sum_{r=0}^{\infty}h_r(t-(\alpha a+(1-\alpha)b))^t$$

If series is convergent in whole real axis.

Definition 2: The series

$$\chi_\alpha(D_\alpha(g),t)=\sum_{r=0}^{\infty}D(g,\alpha;r)(t-t_\alpha)^r$$

is called α -parametrized differential inverse transformation (α -PDIT) of g, if series convergent in the whole real axis. Note: in case $\alpha=0$ and $\alpha=1$ then, $\chi_0(D_0(g),t)=g(t)$ and $\chi_1(D_1(g),t)=g(t)$ are hold, respectively, for any analytic function g. i.e. $\chi_\alpha=D_\alpha^{-1}$. And if $\alpha=0$ and $\alpha=1$, then $\alpha-PDT$ is reduced to at endpoints $t=b$ and $t=a$, respectively. Similarly, in case α -PDIT reduced to DIT. Therefore α -PDTM is generalization of DTM.

Definition 3: Let $n\in \mathrm{N}$ then, the series

$$D_{\alpha,n}(g)=(D(g,\alpha;0),D(g,\alpha;1),...,D(g,\alpha;n),0,0,....)$$

is called n-term α -PDT of g and

$$g_{\alpha,n}(t)=\chi_\alpha(D_{\alpha,n}(g),t)=\sum_{r=0}^{n}D(g,\alpha;r)(t-t_\alpha)^r$$

is called nth-term α - parametrized approximation of g where $t_\alpha=a\alpha+(1-\alpha)b$

Note: Use of n-term α -PDT is convenient in practical applications.

Some Fundamental Operations			
Original Function *g(t)*	Differential Transformed *G(r)*	α-Parametrized differential Transformed $\chi_\alpha(D_\alpha(g),t)$	n-term α-Parametrized differential transformed $\chi_{\alpha,n}(D_{\alpha,n}(g),t), n=0,1,2,...$
$c=\text{constant}$	$g(t)$	$g(t)$	$g(t)$
$\gamma g_1(t),\gamma\in\mathbb{R}$	$\gamma G_1(r)$	$\gamma\chi_\alpha(D_\alpha(g),t)$	$\gamma\chi_{\alpha,n}(D_{\alpha,n}(g),t)$
$g_1(t)\pm g_2(t)$	$G_1(r)\pm G_2(r)$	$\chi_\alpha(D_\alpha(g_1),t)\pm$ $\chi_\alpha(D_\alpha(g_2),t)$	$\chi_{\alpha,n}(D_{\alpha,n}(g_1),t)\pm$ $\chi_{\alpha,n}(D_{\alpha,n}(g_1),t)$

Theorem: Let $f(t)=g^{m}(t), n\in \mathrm{N}.$ then

$$D(f,\alpha;r)=\frac{(r+m)!}{r!}D(f.\alpha;r+m)\ or\ \frac{d^{m}}{dt^{m}}g_{\alpha,n}(t)=\sum_{r=0}^{n}\frac{(r+m)}{r!}D(g,\alpha;r+m)(t-t_{\alpha})$$

Theorem if $g(t)=g_1(t)g_2(t),$ then

$$D(g,\alpha;r)=\sum_{m=0}^{r}[\alpha Y_m(g_1,\alpha)Y_{r-m}(h,a)+(1-\alpha)Y_m(g_2;b)Y_{r-m}(h;b)]$$

Numerical Example

Example 1: Application to BVP let us 3rd order homogeneous differential equation

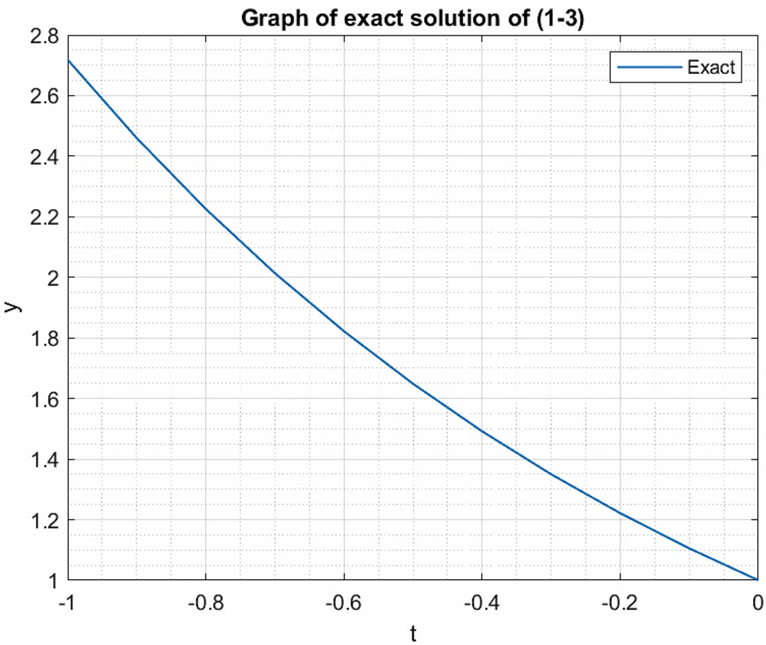
$$y'''(t)+y(t)=0, t\in[-1,0] \tag{1}$$

with non-homogeneous boundary conditions

$$y(-1)=e, y'(0)=-1 and\ y''(0)=1 \tag{2}$$

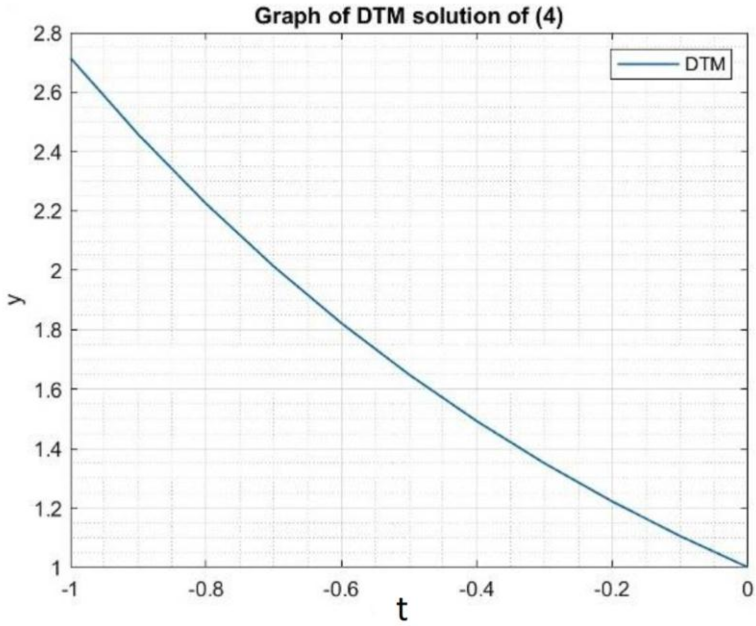
Exact solution and graph of (1) are

$$y(t)=e^{-t} \tag{3}$$



Now, Appling DTM on (1)

$$y(t) = 1 - t + \frac{t^2}{2} - \frac{t^3}{6} + \frac{t^4}{24} - \frac{t^5}{120} - \frac{t^6}{720} + \frac{t^7}{5040} + \frac{t^8}{40320} + \ldots \tag{4}$$



After applied $\alpha - PDT$ on both sides of (1),

$$D(y'''+2y,\alpha;t) = (t+3)(t+2)(t+1)D(y,\alpha;t+3) + D(y,\alpha;t) = 0\ , \tag{5}$$

By definition of $\alpha - PDT$, $y_\alpha(t) = \sum_{r=0}^{\infty} D(y,\alpha;t)(t-t_\alpha)^r$

where $t_\alpha = \alpha a + (1-\alpha)b$,

$$y'_\alpha(t) = \sum_{r=0}^{\infty} r\,D(y,\alpha;r)(t-t_\alpha)^{r-1}\ ,\ y''_\alpha(t) = \sum_{t=0}^{\infty} r(r-1)\,D(y,\alpha;r)(t-t_\alpha)^{r-2}$$

Moreover, for the boundary conditions,

$$y_\alpha(-1) = \sum_{r=0}^{N} D(y,\alpha;r)(-1-(-\alpha))^r = e \tag{6}$$

$$y'_\alpha(0) = \sum_{r=0}^{N} r\,D(y,\alpha;r)(0-(-\alpha))^{r-1} = \sum_{r=0}^{N} r\,D(y,\alpha;r)(\alpha)^{r-1} = -1 \tag{7}$$

$$y''_{\alpha}(0)=\sum_{r=0}^{N} r(r-1)\,D(y,\alpha;r)(0-(-\alpha))^{r-2}=\sum_{r=0}^{N} r(r-1)\,D(y,\alpha;r)(\alpha)^{r-2}=1 \qquad (8)$$

respectively. Let $D(y,\alpha;0)=A, D(y,\alpha;1)=B\,and\,D(y,\alpha;2)=C$ and then putting in the recursive relation (5) $D(y,\alpha;3)=-\frac{A}{6}$ is obtained. Now proceeding the iteration using (5), the other terms of the α −parametrized sequence $D(y,\alpha;n)$ can be calculated as

$$D(y,\alpha;4)=-\frac{B}{24}, D(y,\alpha;5)=-\frac{C}{60}, D(y,\alpha;6)=-\frac{A}{720},$$
$$D(y,\alpha;7)=\frac{B}{5040}, D(y,\alpha;8)=\frac{C}{20160},...$$

Hence, the α −parametrized series solution $y(t,\alpha)$ is evaluated up to $N=8$,

$$y(t,\alpha)=\sum_{r=0}^{8} D(y,\alpha;r)(t-t_{\alpha})^{r}=A+B(t+\alpha)+C(t+\alpha)^{2}-\frac{A}{6}(t+\alpha)^{3}-\frac{B}{24}(t+\alpha)^{4}$$
$$-\frac{C}{6}(t+\alpha)^{5}+\frac{A}{720}(t+\alpha)^{6}+\frac{B}{5040}(t+\alpha)^{7}+\frac{C}{20160}(t+\alpha)^{8} \qquad (9)$$

where $t_{\alpha}=-\alpha$, apply boundary condition we get

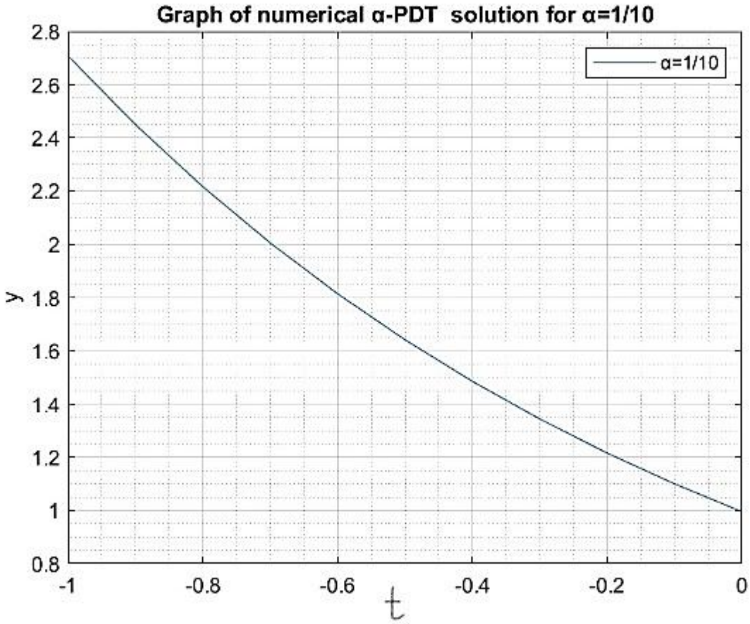
$$y(-1,\alpha)=A+B(\alpha-1)+C(\alpha-1)^{2}-\frac{A}{6}(\alpha-1)^{3}-\frac{B}{24}(\alpha-1)^{4}-\frac{C}{60}(\alpha-1)^{5}$$
$$+\frac{A}{720}(\alpha-1)^{6}+\frac{B}{5040}(\alpha-1)^{7}+\frac{C}{20160}(\alpha-1)^{8}=e$$

$$y(0,\alpha)=B+2C\alpha-\frac{A}{2}\alpha^{2}-\frac{B}{6}\alpha^{3}-\frac{C}{12}\alpha^{4}+\frac{A}{120}\alpha^{5}+\frac{B}{720}\alpha^{6}+\frac{C}{2520}\alpha^{7}=-1$$

and

$$y(0,\alpha)=2C-A\alpha-\frac{B}{2}\alpha^{2}-\frac{C}{3}\alpha^{3}+\frac{A}{24}\alpha^{4}+\frac{B}{120}\alpha^{5}+\frac{C}{360}\alpha^{6}=1\,.$$

Hence, the numbers A,B, and C are calculated from the boundary conditions(2) and used to draw the graph at various values of α.


Graph of numerical α-PDT solution for α=1/10
α=1/10
y
t
2.8
2.6
2.4
2.2
2
1.8
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1.4
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1
0.8
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-0.6
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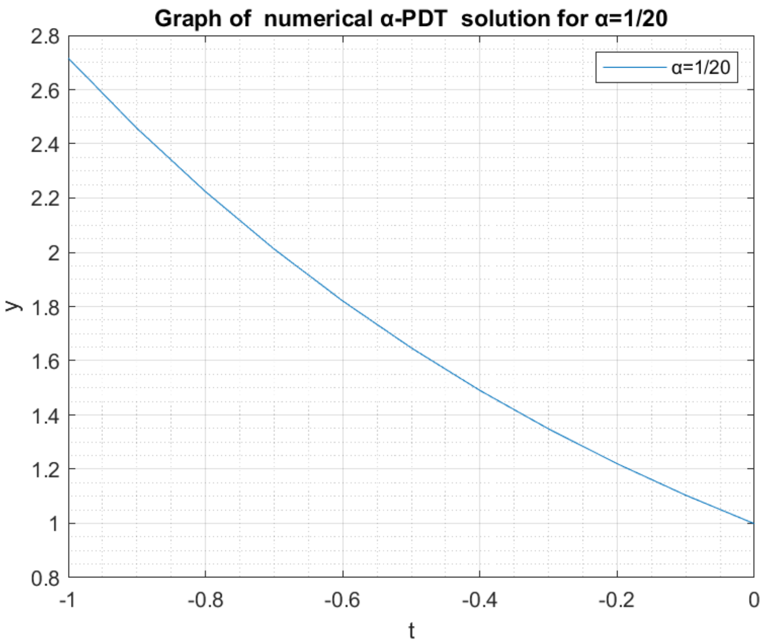
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α=1/20
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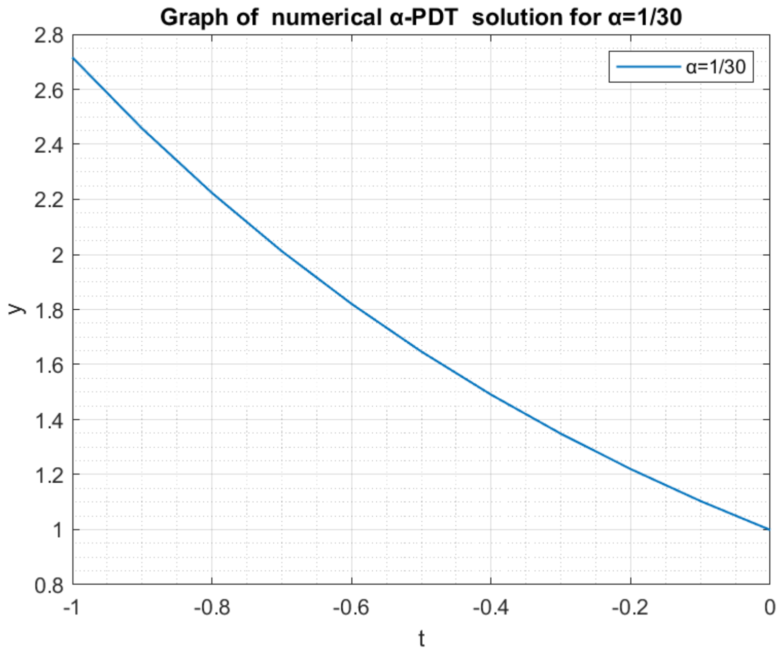
Graph of numerical α-PDT solution for α=1/30
α=1/30
y
t
2.8
2.6
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-1
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-0.6
-0.4
-0.2
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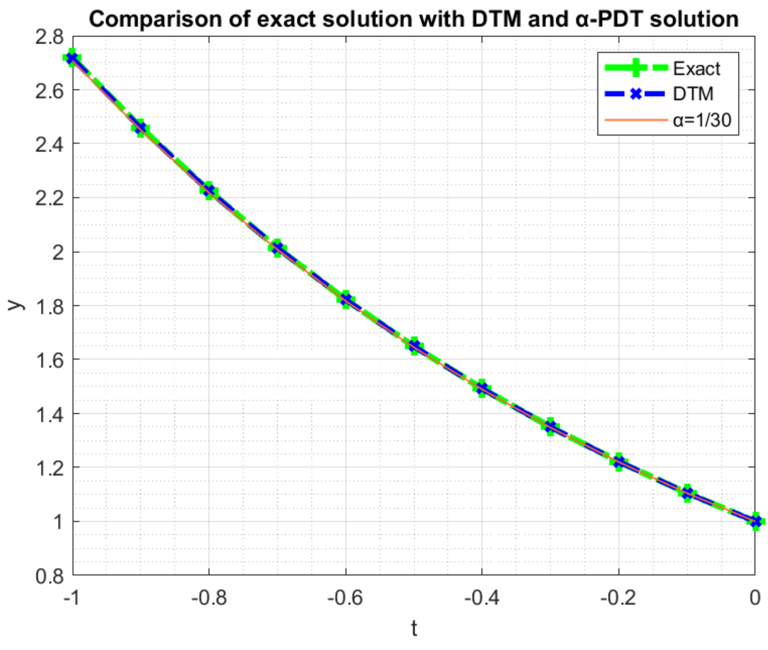

Comparison of exact solution with DTM and α-PDT solution
Exact
DTM
α=1/30
y
t
2.8
2.6
2.4
2.2
2
1.8
1.6
1.4
1.2
1
0.8
-1
-0.8
-0.6
-0.4
-0.2
0

Example 2: Let us 4th order non-homogeneous differential equation

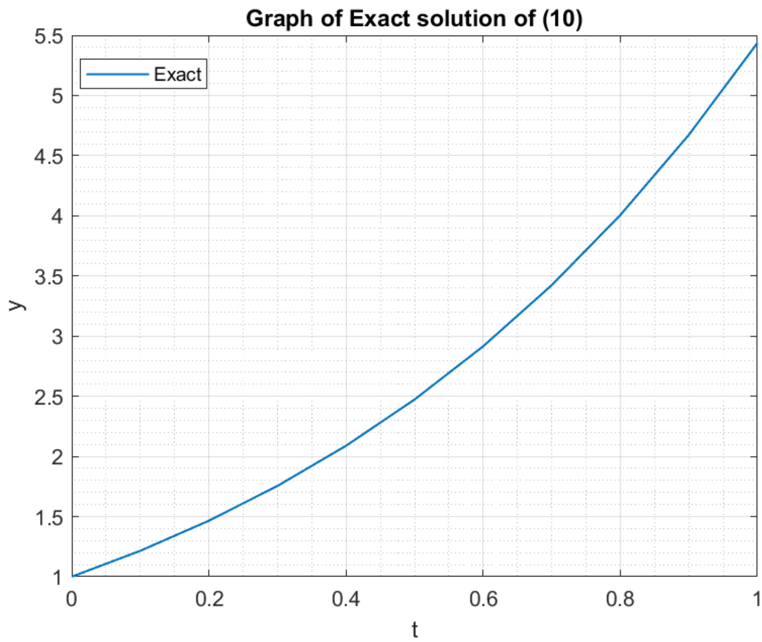
$$y^{iv}(t) - y(t) = 4e^{t}, \ t \in [0,1] \tag{10}$$

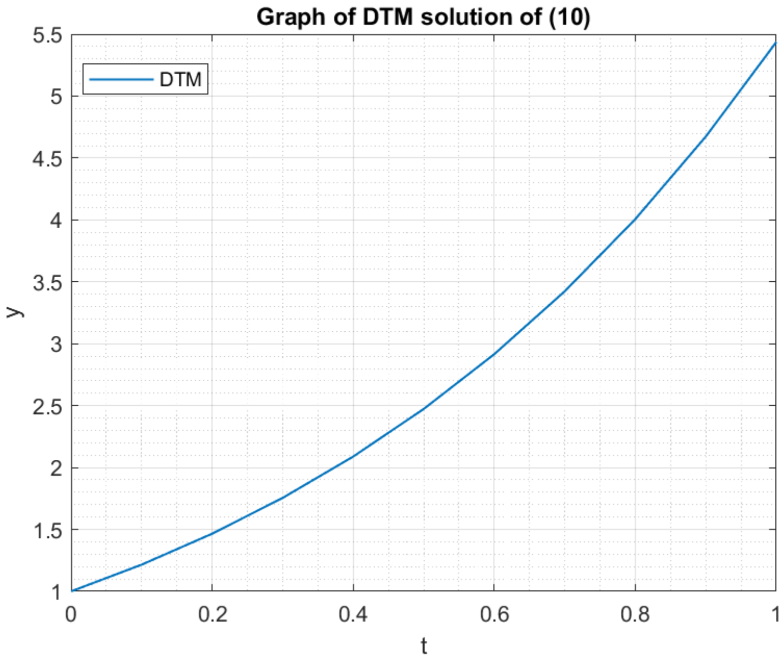
with boundary conditions,

$$y(0) = 1, y''(0) = 3, y(1) = 2e \, and \, y''(1) = 4e \tag{11}$$

The exact-solution for this problem:

$$y(t) = (1+t)e^{t} \tag{12}$$





If it is applied α -PDT to both sides of, then:

$$(r+4)(r+3)(r+2)(r+1)D(y,\alpha;r+4)=\frac{4}{r!}+D(y,\alpha;r) \tag{13}$$

Therefore, form the definition of

$$\alpha - PDT,\ y_\alpha(t)=\sum_{r=0}^{\infty}D(y,\alpha;k)(t-t_\alpha)^r \tag{14}$$

Moreover, for boundary conditions

$$y_\alpha(0)=\sum_{r=0}^{N}D(y,\alpha;r)(0-(1-\alpha))^r=1,$$

$$y''_\alpha(0)=\sum_{r=0}^{N}D(y,\alpha;r)(0-(1-\alpha))^r=3,\ y_\alpha(1)=\sum_{r=0}^{N}D(y,\alpha;r)(1-(1-\alpha))^r=2e$$

and

$$y''_\alpha(1)=\sum_{r=0}^{N}D(y,\alpha;r)(1-(1-\alpha))^r=4$$

Respectively, now, denoting $D(y,\alpha;0)=A_1, D(y,\alpha;1)=A_2, D(y,\alpha;2)=A_3$ and $D(y,\alpha;3)=A_4$ and the substituting in recursive relation, $D(y,\alpha;4)=\dfrac{4+A_1}{24}$,

$D(y,\alpha;5)=\frac{4+A_2}{120}$, $D(y,\alpha;6)=(2+A_3)/360$, $D(y,\alpha;7)=(2/3+A_4)840$,

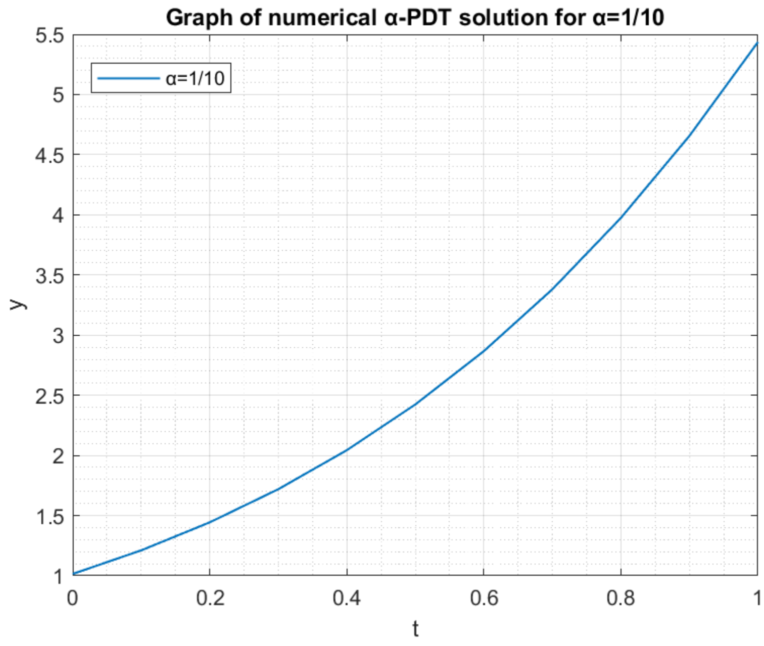
$D(y,\alpha;8)=\frac{1}{1680}\left(\frac{1}{3}+\frac{A_1}{24}\right)$, $D(y,\alpha;9)=\frac{1}{3024}\left(\frac{1}{15}+\frac{A_2}{120}\right)$,

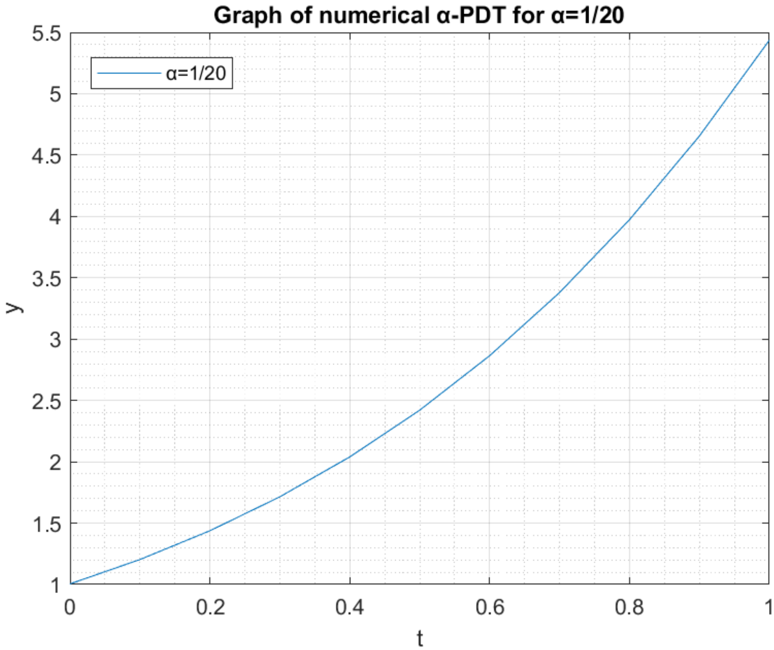
$D(y,\alpha;10)=\frac{1}{5040}\left(\frac{1}{90}+\frac{A_3}{360}\right)$ and $D(y,\alpha;11)=\frac{1}{7920}\left(\frac{1}{630}+\frac{A_4}{840}\right)$ then α -PDT series solution $y(t,\alpha)$ is evaluated up to N=11;

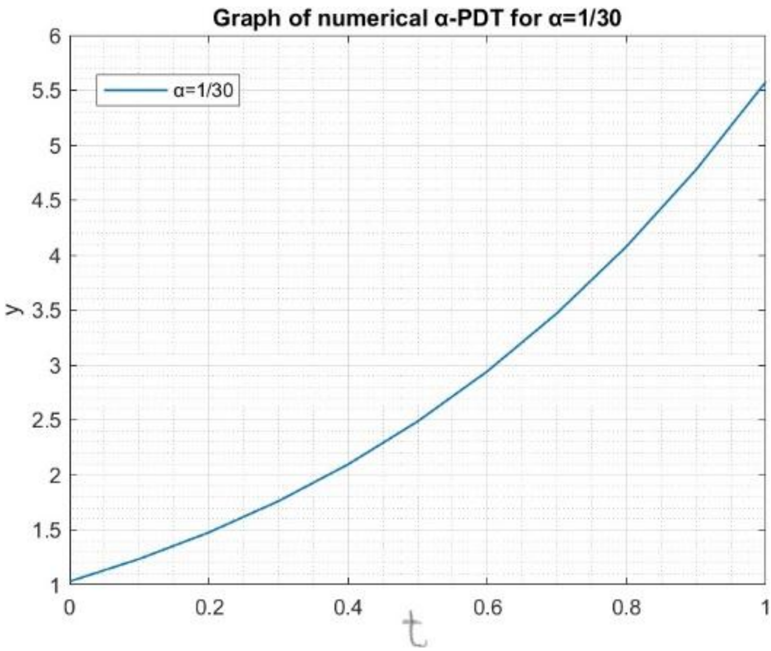
$$y(t,\alpha)=\sum_{N}^{11}D(y,\alpha;11)(t-t_\alpha)^r \tag{15}$$

$$\begin{aligned}&=A_1+A_2(t+\alpha-1)+A_3(t+\alpha-1)^2+A_4(t+\alpha-1)^3+\left(\frac{4+A_1}{24}\right)(t+\alpha-1)^4\\
&\quad+\left(\frac{4+A_2}{120}\right)(t+\alpha-1)^5+\left(\frac{2+A_3}{360}\right)(t+\alpha-1)^6+\left(\frac{2}{3}+A_4\right)\frac{(t+\alpha-1)^7}{840}\\
&\quad+\left(\frac{1}{3}+\frac{A_1}{24}\right)\frac{(x+\alpha-1)^8}{1680}+\left(\frac{1}{15}+\frac{A_2}{120}\right)\frac{(t+\alpha-1)^9}{3024}+\left(\frac{1}{90}+\frac{A_3}{360}\right)\frac{(t+\alpha-1)^{10}}{5040}\\
&\quad+\left(\frac{1}{630}+\frac{A_4}{840}\right)\frac{(t+\alpha-1)^{11}}{7920}+\dots\end{aligned}$$

For more $A_{1,}A_{2,}A_2$ and A_4 are calculated from boundary conditions (11) and numerical $\alpha-PDT$ solutions are presented at different values of α.




Graph of numerical α-PDT for α=1/20
α=1/20
y
t
5.5
5
4.5
4
3.5
3
2.5
2
1.5
1
0
0.2
0.4
0.6
0.8
1


Graph of numerical α-PDT for α=1/30
α=1/30
y
t
6
5.5
5
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3.5
3
2.5
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1.5
1
0
0.2
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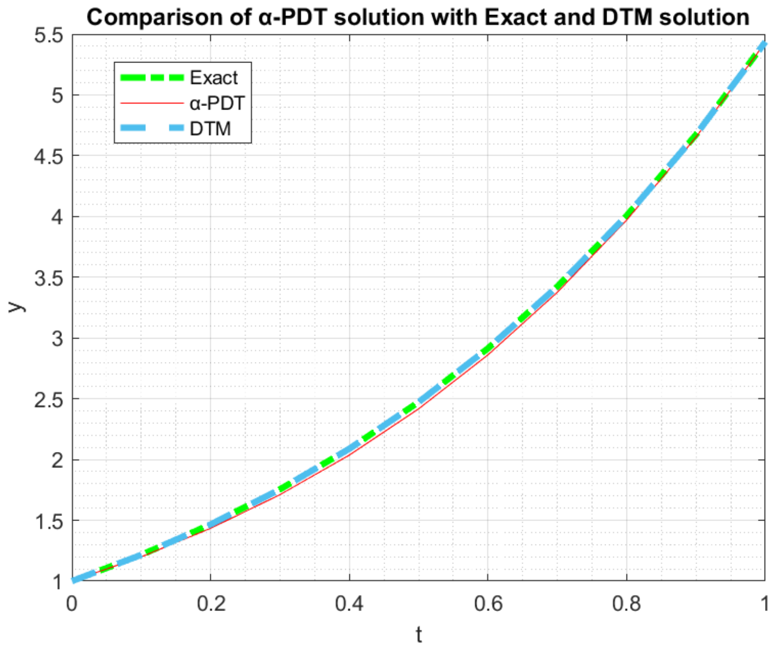
0.8
1



CONCLUSION

In this article, the new generalization of differential-transformation method (DTM), that is α-Parameterized differential-transformation method (α-PDTM) has applied to solve homogenous and non-homogenous higher-order boundary-value problems. Two examples are solved which showed that approximate solutions obtained through α-Parameterized differential-transformation method (α-PDTM) is efficient, accurate and more reliable and easier to apply as compared to exact-solutions and differential transformation method. This technique is much simple and effective as it is based on straight-forward steps and offers a lot of potential in science and engineering towards higher order problem.

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STABILITY OF VELOCITY OF MONEY IN PAKISTAN

Abida Hafeez[1], Sahar Latif Rana[2], Nadia Hanif[3] and **Noman Arshed[4]**
Division of Management and Administrative Science
University of Education, Lahore, Pakistan
Email: [1]abida.hafeez@ue.edu.pk
[2]nadia.hanif@ue.edu.pk
[3]sahar.latif@ue.edu.pk
[4]noumanarshad@hotmail.com

ABSTRACT

The purpose of this study is to examine the stability of the velocity of money in Pakistan. The data set contains the time period from 1960 to 2020. The methodology adopted in this paper contains Dynamic Ordinary Least Square (DOLS) and Stability tests (Chow test, CUSUM and CUSUM Squares). Theoretically and empirically, permanent income (YP) behaves positively exhibiting that the real permanent income will lead to raising the number of transactions hence positively affecting velocity in the economy of Pakistan. Similarly, real rate of interest is appeared to hold a positive relationship with velocity. The findings show that permanent income significantly and positively affects velocity. On the other hand, cycle/transitory income appears positive having a value of less than one showing that velocity is pro-cyclical and confirms the permanent income hypothesis of Friedman. The results of stability tests indicate that velocity of money is not stable in Pakistan as a result of changes in regulatory framework and financial reforms.

KEYWORDS

Velocity, Permanent Income Hypothesis, DOLS, Money, Pro-cyclical.

INTRODUCTION

Choice of monetary policy regime is based on stability of velocity of money. Central bank can adopt either monetary targeting strategy or inflation targeting strategy. A stable money demand function is a pre-requisite for central bank's strategy of monetary targeting in turn it leads to stability in velocity of money in the economy. On contrary, unstable velocity is expected to emerge from interest rate volatility (Omer, 2010). In addition, an unstable velocity not only lead to unstable demand function of money but also affirm the viewpoint of Omer & Saqib (2009) regarding use of central banks' strategy of inflation targeting instead of monetary intermediate targeting.

In the context of money demand, the main source of instability is interest rate. The volatility in rate of interest causes the instability in velocity, thereby, leads to unstable the money demand. However, change in definition of development of various financial institutions or money is another cause of instability (Bordo & Jonug, 2004).

Monetary intermediate targeting is relied on a predictable as well as stable relationship between monetary aggregates and inflation. However, Moinuddin, (2009) highlights that financial innovation, increasing use of technology, and structural changes in financial sector and economy have a significant effect while weakening the relationship of money and inflation. A stable money demand function is explored in India in wake of uncertainty in stock market and in economic and monetary systems (Khan, et al., 2021). The objective of this study is to examine the stability of velocity of money in Pakistan. During 1990s, various measures were adopted for the development/strengthen financial institutions, improvement in domestic debt management, changes in banking rules and regulations, regulatory changes and liberalisation in capital market as well as in foreign exchange. Foreign exchange market's activities were speedily flourished after allowing opening foreign currency accounts to residents along with money changers were granted licenses. Moreover, activities of capital market were also expanded as result of opening up stock markets for foreign investors. Trinh (2022) highlights the importance of velocity money while expanding money market structure.

The characteristics of the economy of Pakistan's economy changed after these financial and institutional developments particularly, with respect to financial sector. Consequently, there appeared more financial innovations, products, intermediaries and banks. If velocity of money appears unstable indicating that demand function is no more stable. Consequently, monetary intermediate target is not a viable strategy. Therefore, the SBP can adopt inflation targeting as an alternative.

Hypothesis

H_0: Velocity of money is stable over time

H_1: Velocity of money is not stable over time

Research Question

How can SBP achieve target in case the velocity of money is not stable?

LITERATURE REVIEW

Money demand has been explored via 'conventional' and 'velocity' formulation in literature. This section mainly focuses literature related to velocity. The drawbacks of conventional money demand function heightened during 1980s because velocity (PY/M) has been underestimated as it grew faster than expected.

During 1980s, many economists show their concern about declining the velocity monetary aggregates, particularly M1. The fundamental reason explored by economists was the continuous use of monetary targets by the central banks. Since 1982, one of the key reasons given by policymakers to abandon monetary intermediate targets in USA and other economies is the unpredictability of velocity. The vital role of velocity in the stability of money prompted various researchers for conducting studies regarding velocity. Siklos (1993) states that money demand function (MDF) can be explored through velocity. In developed economies, behaviour of velocity is explored via long term data using the econometric technique such as ordinary least square (OLS) and it has been experienced that velocity declined during the period of monetization and it was recovered through financial innovation and deregulation (Bordo & Jonug, 1981, 1987, 1990 & 2004). Various developed economies such as Switzerland, UK and Germany found a long-term and stable relationship between money and its determinants. Hamori & Hamori (1999) found unification in Germany during 1990 for structural break causing instability. Similarly, developing countries such as Nepal and China experienced stable money demand. In case of China, rapid changes in financial developments led to instable money demand during 1980s. Moreover, structural breaks explored during 1980 and 1993 due to decisive economic and financial developments by Lee & Chien (2008).

Bilquees & Shehnaz (1994) explained the slowdown in velocity of money during 1974-75 to 1991-92 in Pakistan and highlighted that velocity of money was significantly affected by financial development for which number of bank branches used as a proxy. Omer & Saqib (2009) argued that monetary targeting strategy of SBP was not favourable one due to instable money velocity. They highlighted that income velocities of M0, M1 and M2 were not constant in Pakistan as assumed by quantity theory of money (QTM) and these three velocities were unstable.

Moinuddin (2009) indicated a negative intercept estimated model of broad money. However, no satisfactory explanation for large intercepts was given that ultimately lead to specification bias. For instance, Bordo & Jonung (1990) proposed that expected inflation should be a part of demand function in such (regulated) economies where rate of interest was not free while responding market forces. On the other hand, Moinuddin (2009) covered the time period from 1975 to 1991 when economy experienced regulations to the greater extent.

Different studies regarding stability of money used the sample till 2000 and their samples did not incorporate the second generation of financial reforms. Therefore, the findings of these studies were unable to explain the effects of financial reforms (Bahmani-Oskooee & Rehman, 2005; and Qayyum, 2006). Another study shed light on stable and long run M2 demand function through the significance of the results but did not estimate demand function via stability test (Abbas & Husain, 2006).

Khan, et al. (2021) addressed instability in money demand due to uncertainty drom 2003 to 2019 using ARDL model. The results indicate that there exists a stable money demand function in India in wake of uncertainty in stock market and in economic and monetary systems. Zehra et al. (2021) explored the money demand under the umbrella of macroeconomic variables from 1970 to 2018 using both ARDL and NARDL models. The findings indicate that inflation is negatively related to money demand both in long run and short run. The stability tests (CUSUM and CUSUM square) have verified the stability of broad money (M2) demand money in case of Pakistan.

The literature reviewed in this study so far reflected the extensive use of conventional formulation of money demand function owing to simple interpretation and analytical formulation. Arango & Nadiri (1981)'s model was being used to estimate conventional money demand function and this model was seriously criticized due to lack of theoretical basis. However, study conducted by Omer (2010) used velocity formulation of money demand to investigate the velocity of money. In addition, the velocity formulation had strong theoretical basis and can be explained through economic theory proposed by Milton Friedman i.e. Permanent Income Hypothesis (YP).

DATA AND METHODOLOGY

Data on different variables such as CPI, M2, per capita income (PCI), and Nominal and real income is extracted from the databank of World Development Indicators (WDI). Data on interest rate is collected from International Financial Statistics (IFS). A series of expected inflation is generated on the basis of adaptive expectations in the form of p(-1). The annual data has been used from 1960 to 2020 to examine the stability of velocity. In this study, only one version of velocity out of three (Omer, 2009) is used. Only M2 is taken as monetary aggregate from three available monetary aggregates and its velocity. In addition, M2 is being used as an intermediate target in this study. All variables are estimated in logarithmic except interest rate and the inflation. The variable, real per capita permanent income computed through applying Bordo & Jonug (1990)'s HP filter (λ=100) to find long-term trend of per capita real GDP since 1960. The Chow test has been used to find out the stability of structural parameters. Tests such as CUSUM and CUSUM Squares are also applied to find out stability.

The ADF test is employed to find out the stationary of variables, which is estimated through following equation:

$$\Delta Y_t = \beta_1 + \beta_{2t} + \alpha_0 Y_t - 1 + \alpha_i \sum_{i=1}^{m} \Delta Y_t - i + \varepsilon_t \tag{1}$$

The Alternative hypothesis can be accepted as α_0 appears statistically significant.

The following econometric model is estimate through Dynamic Ordinary Least Square (DOLS) to find the long-term relationship among the variables. The model is as follows:

$$\log(V)_t = \alpha_0 + \beta_1 \log(YP) + \beta_2 \log(Cycle) + \beta_3 R + \beta_4 P(-1) + \varepsilon_t \tag{2}$$

where, V stands for velocity of money. YP represents permanent income and Cycle for transitory income. R is interest rate and P(-1) presents a series of expected inflation on the basis of adaptive expectations.

RESULT DISCUSSION

This section presents the empirical results. The results are presented in tabulated form to find the stability of the velocity of money in case of Pakistan from 1960 through 2020.

Table 1
Unit Root Analysis

Variables	ADF				Remarks
	Level	Prob.	I^{st} Diff.	Prob.	
Velocity	**5.3345**	**1.0000**	**-4.2380**	**0.0001*** **	**I(1)**
PY	**2.4694**	**0.9964**	**-3.2383**	**0.0016*** **	**I(1)**
Cycle	**-4.6244**	**0.0000*** **	**-**	**-**	**I(0)**
R	**-0.5240**	**0.4857**	**-6.3422**	**0.0000*** **	**I(1)**
P	**0.9238**	**0.9032**	**-7.4642**	**0.0000*** **	**I(1)**

Note: Rejection of the null hypothesis (the variable has unit root) at 1 percent level is indicated by three asterisk (***), or 5 percent level by two asterisk (**) or, 10 percent level by one asterisk (*).

The findings show that the variables are integrated of different orders. Using an augmented Dickey–Fuller (ADF) test, all the variables are stationery at first difference except Cycle, which is stationery at level and decision is based on 5% level of significance.

Table 2
Results of Dynamic Ordinary Least Squares (DOLS)
Dependent Variable: LOG (V)

Variable	Slope Coefficient	Prob.
LOG(YP)	**0.8924**	**0.0314** **
LOG(CYCLE)	**0.0388**	**0.8391**
R	**0.1357**	**0.0091*** **
P(-1)	**0.0074**	**0.0000*** **
	R-squared	**0.6847**

Note: 1 percent level is indicated by three asterisk (***), or 5 percent level by two asterisk (**) or, 10 percent level by one asterisk (*).

Theoretically, permanent income (YP) is expected to be positively related to velocity, showing that a rise in permanent income (YP) leads to a rise in the number of transactions in an economy, hence, have a positive effect on the velocity. The results indicate that permanent income affects velocity positively and significantly.

The 'Cycle' or transitory income should have a unity as a value of coefficient in DOLS regression. The value of coefficient of cycle contains positive but less than one, which indicates that the velocity explains the pro-cyclically movement and consistent with the permanent income hypothesis of Friedman. The transitory income leads to increase the money demand over the cycle as cash balance serves as a buffer. However, these transitory balance(s) will be worked off, while bringing back to a unity coefficient in the long run (Bordo & Jonug, 1990).

The real interest rate (R) is also showing a positive sign like permanent income (YP). Theoretically, an increase in real interest rate will lead to a decline in real money demand/balances, thereby, leading money velocity (PY/M) to rise. The effect of inflation on velocity appears ambiguous theoretically and the sign of coefficient may be negative or positive. As far as the selection of the year for structural break is concerned, 1991 has been selected to check the stability of the parameters. The year 1991 is taken as a structural break as most of the structural and fundamental changes in the financial sector of Pakistan and a reform package were introduced during this period. Chow break-point test has been suggested as the source of structural break(s) Gujrati (2003). The value of F-statistics is 153.80 with p-value (0.000) indicating that there exists a structural break reflecting that velocity of money is not stable over the time period.

Table 3
Application of the Chow Test/Stability Test

F-statistic	**153.7976**	**Prob.**	**0.0000**

Stability test (CUSUM and CUSUM Squares): First one is showing stability and second one is unstable. The stability test (CUSUM) in figure indicates that velocity of money is stable from 1960 to 2020. However, stability test (CUSUM Squares) in figure 2 shows that velocity of money is not stable reflecting structural changes in money demand function.

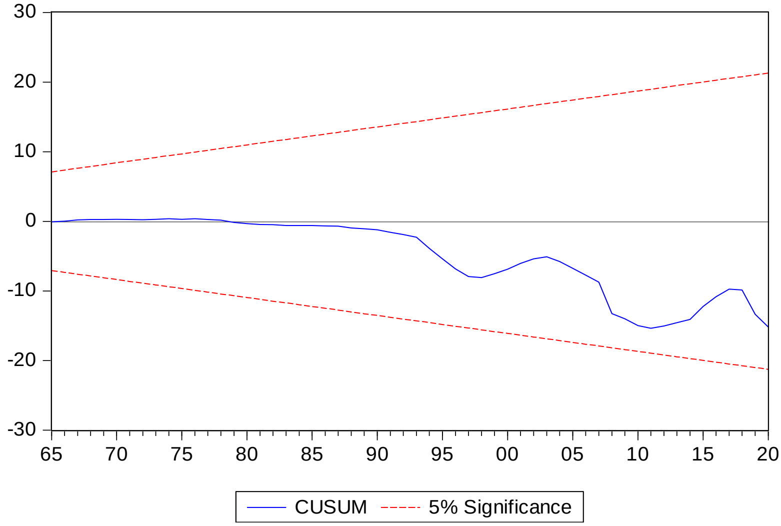

Figure 1: Stability Test (CUSUM)

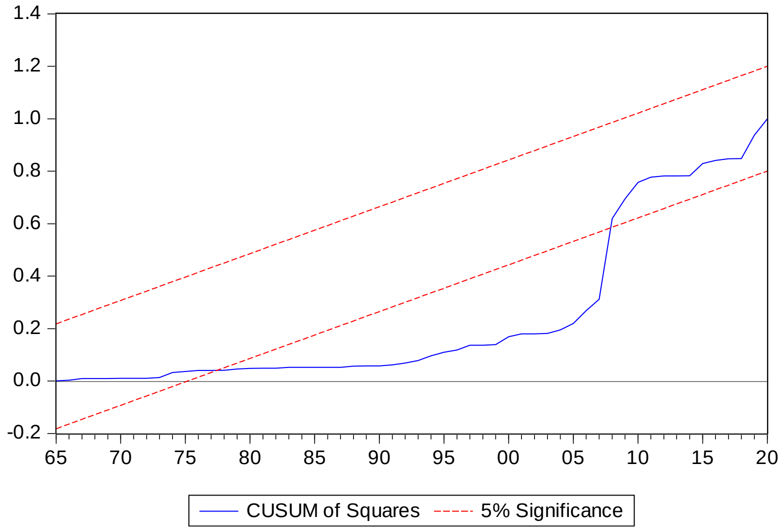

Figure 2: Stability Test (CUSUM Squares)

CONCLUSION

The purpose of this study is to examine the stability of the velocity of money in Pakistan from 1960 to 2020. The research methods adopted includes Dynamic Ordinary Least Square (DOLS) as well as stability tests (Chow test, CUSUM and CUSUM Squares). Theoretically and empirically, permanent income (YP) behaves positively exhibiting that the real permanent income will lead to raising the number of transactions hence positively affecting velocity in the economy of Pakistan in the long run. Similarly, real interest rate also has a positive relationship with velocity. The findings show that permanent income significantly and positively affects velocity. On the other hand, cycle/transitory income appears positive having a value of less than one showing that velocity is pro-cyclical and confirms the permanent income hypothesis of Friedman. The results of stability tests (Chow test and CUSUM Squares) indicate that velocity of money is not stable in Pakistan due to changes in regulatory framework and financial reforms. Consequently, monetary intermediate target is not a viable strategy. Therefore, the SBP can adopt inflation targeting as an alternative.

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A STATISTICAL ANALYSIS OF THE LARGE-SCALE MANUFACTURING INDUSTRY GROWTH OF PAKISTAN

Adeel Ahmed[1] and **Muhammad Adil**[2]

[1] School of Natural Sciences, National University of Sciences and Technology, Islamabad, Pakistan.
Email: aahmed.msstat20sns@student.nust.edu.pk

[2] Pakistan Bureau of Statistics, Islamabad, Pakistan
Email: adil.so.fbs@gmail.com

ABSTRACT

The LSM (Large Scale Manufacturing) industry is one of the three major components of economic growth of a country, other two being Agriculture and Services. Through the course of time, Pakistan's LSM industry has seen about its fair share of ups and downs but so far, no concrete study has been performed to study this sector. This paper intends to investigate the behavior of the LSM industry of Pakistan for the time period 2012 to 2021 and to put light on this sector's future. The study would contribute to letting us know which factors (or variables) are more significantly impacting the growth of the LSM industry in Pakistan, secondly which statistical model best describes the relationship between the variables. For this study, time series data (from year 2012) has been obtained from the online archives/repositories of Pakistan Bureau of Statistics, State Bank of Pakistan, and Pakistan Economic Surveys. After stationarity checks and lag selection, the data is estimated by VAR (vector autoregression) and SVAR (structural vector autoregression) models. Next, we look at the IRF (impulse response function) for the variables keeping LSM/Quantum Index of Manufacturing (percentage change or growth of QIM) as response, and lastly, the data has been forecasted for the next 36 months. This study may help the policy makers as well as scientists in future planning regarding allocation of national resources in the right sub-sectors of economy.

KEYWORDS

LSM, Pakistan, QIM, VAR, SVAR, IRF.

1. INTRODUCTION

The inspiration that lead to this study is to analyze and understand the behavior of the large scale manufacturing industry of Pakistan. As mentioned above manufacturing is one of the three predictors of a country's growth (other two being Agriculture and Services).

Manufacturing sector can be further classified into two categories namely the large-scale manufacturing and the small-scale manufacturing. We are focusing on the large-scale manufacturing sector as it is the major component of manufacturing sector.

Our goal being to identify the predictors/variables the affect the growth of the large-scale manufacturing.

So far very little (recent) work has been done in this area of Pakistan's economy. AR Kemal tried to establish a consistent time series data relating to the large-scale manufacturing industry of Pakistan in 1976 (Kemal, 1976). S. Wizarat explored the sources of growth in the large-scale manufacturing of Pakistan in 1984 (Wizarat, 1984). U. Afridi and M. Aziz investigated the dynamics of change in Pakistan's largescale manufacturing sector in 1985 (Afridi, 1985). Rukhsana Kalim studied the capital intensity in the large-scale manufacturing of Pakistan in 2001 (Kalim, 2001). Tarik, Ejaz and Musleh looked up the efficiency of large-scale manufacturing in Pakistan by a production frontier approach (Mahmood, Ghani & Din, 2006). In 2013 Kathuria and Natarajan checked if Manufacturing is an engine of growth in India (Kathuria & Natarajan, 2013). Zara, Aman & Muhammad observed Determinants of Industrial Growth in South Asia (Ejaz, Aman Ullah & Khan, 2017).

2. DATA AND METHODOLOGY

For this study we collected monthly data from the online repositories of Pakistan Bureau of statistics ("Pakistan Bureau of Statistics", 2022), State Bank of Pakistan ("State Bank of Pakistan", 2022), and Pakistan Economic Surveys ("| Ministry of Finance | Government of Pakistan |", 2022). Percentage change or growth of Quantum Index of Manufacturing is the dependent variable for our study and percentage change or growth of 20 independent variables that are supposed to be effecting the growth of QIM, namely balance of trade, chemicals, coke and petroleum products, electronics, consumer price index, engineering products, fertilizers, foreign direct investment, food beverages and tobacco, gross domestic product, iron and steel, leather products, paper and board, nonmetallic mineral products, pharmaceuticals, textile, wood products, wholesale price index, rubber products, automobiles.

After stationarity checks using augmented Dickey Fuller test and lag selection, the data is estimated by VAR (vector autoregression) and SVAR (structural vector autoregression) models. Next, we look at the IRF (impulse response function) for the variables keeping QIM as response. It is worth noting here that "automobile industry" was not stationary at first or second difference, so we removed it from our model.

To validate the IRFs we observed the FEVD (Forecast Error Variance Decomposition) plots. And lastly, the data has been forecasted for the next 36 months.

This is the model for our estimation:

$$\begin{aligned}
QIM = {} & QIM.l1 + GDP.l1 + FDI.l1 + CPI.l1 + WPI.l1 + dBOT.l1 + FBT.l1 \\
& + dTEXTILE.l1 + dCOKEPETROLEUM.l1 + dNONMETALLIC.l1 \\
& + ddIRONSTEEL.l1 + dELECTRONICS.l1 + RUBBER.l1 + WOOD.l1 \\
& + FERTILIZERS.l1 + dCHEMICALS.l1 + dPHARMACEUTICALS.l1 \\
& + LEATHER.l1 + ENGINEERING.l1 + dPAPERBOARD.l1 + QIM.l2 \\
& + GDP.l2 + FDI.l2 + CPI.l2 + WPI.l2 + dBOT.l2 + FBT.l2 \\
& + dTEXTILE.l2 + dCOKEPETROLEUM.l2 + dNONMETALLIC.l2 \\
& + ddIRONSTEEL.l2 + dELECTRONICS.l2 + RUBBER.l2 + WOOD.l2 \\
& + FERTILIZERS.l2 + dCHEMICALS.l2 + dPHARMACEUTICALS.l2 \\
& + LEATHER.l2 + ENGINEERING.l2 + dPAPERBOARD.l2 + QIM.l3 \\
& + GDP.l3 + FDI.l3 + CPI.l3 + WPI.l3 + dBOT.l3 + FBT.l3 \\
& + dTEXTILE.l3 + dCOKEPETROLEUM.l3 + dNONMETALLIC.l3 \\
& + ddIRONSTEEL.l3 + dELECTRONICS.l3 + RUBBER.l3 + WOOD.l3 \\
& + FERTILIZERS.l3 + dCHEMICALS.l3 + dPHARMACEUTICALS.l3 \\
& + LEATHER.l3 + ENGINEERING.l3 + dPAPERBOARD.l3 + QIM.l4 \\
& + GDP.l4 + FDI.l4 + CPI.l4 + WPI.l4 + dBOT.l4 + FBT.l4 \\
& + dTEXTILE.l4 + dCOKEPETROLEUM.l4 + dNONMETALLIC.l4 \\
& + ddIRONSTEEL.l4 + dELECTRONICS.l4 + RUBBER.l4 + WOOD.l4 \\
& + FERTILIZERS.l4 + dCHEMICALS.l4 + dPHARMACEUTICALS.l4 \\
& + LEATHER.l4 + ENGINEERING.l4 + dPAPERBOARD.l4.
\end{aligned}$$

3. GRAPHS AND RESULTS

Data Figure

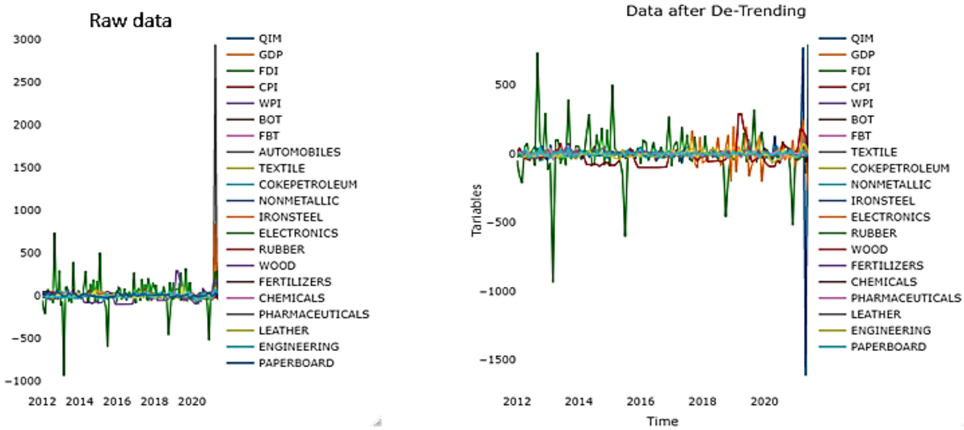

Table 1

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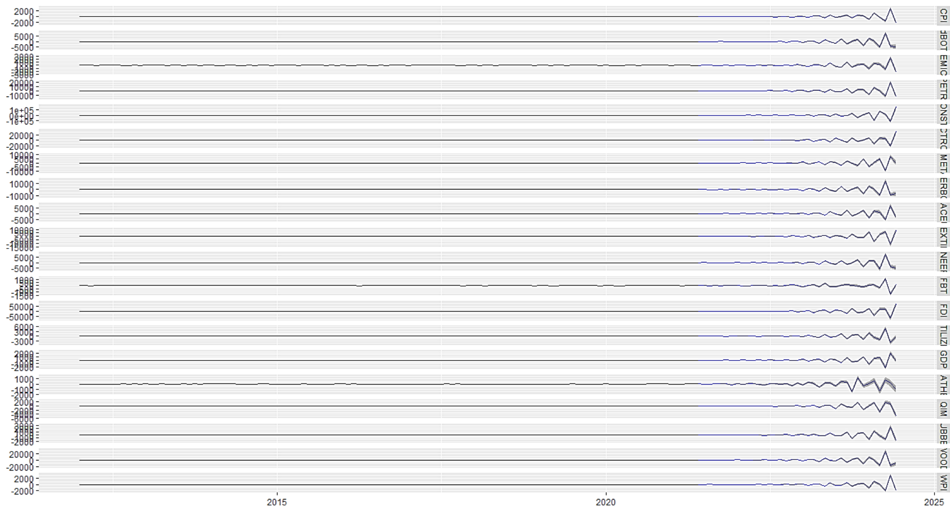
	Estimate	Std. Error	t value	Pr(>|t|)	
QIM.11	0.5954542	0.2687789	2.215	0.05397	.
GDP.11	0.8832187	0.9460822	0.934	0.37490	
FDI.11	0.0225638	0.0181706	1.242	0.24570	
CPI.11	5.5913816	2.5098657	2.228	0.05289	.
WPI.11	-5.3463126	2.4515387	-2.181	0.05710	.
BOT.11	-0.3653246	0.2536481	-1.440	0.18365	
FBT.11	-0.0157880	0.2497164	-0.063	0.95097	
TEXTILE.11	1.4430730	0.9043801	1.596	0.14503	
COKEPETROLEUM.11	-0.0075194	0.3398287	-0.022	0.98283	
NONMETALLIC.11	-0.9139080	0.3673629	-2.488	0.03455	*
IRONSTEEL.11	-0.2304671	0.1573775	-1.464	0.17712	
ELECTRONICS.11	0.0234373	0.0579958	0.404	0.69556	
RUBBER.11	0.5794068	0.4285711	1.352	0.20939	
WOOD.11	0.0767168	0.0753935	1.018	0.33546	
FERTILIZERS.11	-0.4164413	0.2936809	-1.418	0.18987	
CHEMICALS.11	0.2036664	0.5095075	0.400	0.69867	
PHARMACEUTICALS.11	-0.0991659	0.4564001	-0.217	0.83284	
LEATHER.11	-0.1137563	0.1290528	-0.881	0.40099	
ENGINEERING.11	0.0465280	0.2093796	0.222	0.82910	
PAPERBOARD.11	-0.1152719	0.2182362	-0.528	0.61014	
QIM.12	0.3818192	0.2315782	1.649	0.13360	
GDP.12	1.5202758	0.7612426	1.997	0.07691	.
FDI.12	-0.0492118	0.0202567	-2.429	0.03802	*
CPI.12	-0.1618975	3.5945098	-0.045	0.96506	
WPI.12	0.3798839	3.3939857	0.112	0.91334	
BOT.12	-0.0588009	0.2260599	-0.260	0.80063	
FBT.12	-0.6256031	0.2701134	-2.316	0.04578	*
TEXTILE.12	0.0125587	1.1881128	0.011	0.99180	
COKEPETROLEUM.12	-0.3755941	0.3330894	-1.128	0.28865	
NONMETALLIC.12	-0.7519304	0.3590731	-2.094	0.06575	.
IRONSTEEL.12	-0.2909386	0.2362711	-1.231	0.24939	
ELECTRONICS.12	-0.0370848	0.1024437	-0.362	0.72570	
RUBBER.12	-0.7574549	0.3737292	-2.027	0.07332	.
WOOD.12	-0.2248975	0.1051254	-2.139	0.06109	.
FERTILIZERS.12	0.2439277	0.3126080	0.780	0.45525	
CHEMICALS.12	1.3720853	0.6495040	2.113	0.06381	.
PHARMACEUTICALS.12	0.4160672	0.4654795	0.894	0.39468	
LEATHER.12	0.2248144	0.1534121	1.465	0.17685	
ENGINEERING.12	-0.1248153	0.1477775	-0.845	0.42021	
PAPERBOARD.12	0.3264464	0.1868802	1.747	0.11462	
QIM.13	-0.5797039	0.2616872	-2.215	0.05398	.
GDP.13	0.2855627	0.9021041	0.317	0.75880	
FDI.13	-0.0048726	0.0147607	-0.330	0.74887	
CPI.13	-0.5831256	4.4892650	-0.130	0.89951	
WPI.13	0.8801205	4.3119064	0.204	0.84281	
BOT.13	-0.0931744	0.2220931	-0.420	0.68467	
FBT.13	0.7436782	0.3102949	2.397	0.04012	*
TEXTILE.13	1.1290374	0.7182509	1.572	0.15042	
COKEPETROLEUM.13	0.2429557	0.2901925	0.837	0.42414	
NONMETALLIC.13	-0.5776569	0.3211747	-1.799	0.10563	
IRONSTEEL.13	-0.2453682	0.2788158	-0.880	0.40173	
ELECTRONICS.13	-0.1321056	0.1234531	-1.070	0.31244	
RUBBER.13	0.1694446	0.2844637	0.596	0.56608	
WOOD.13	0.2716404	0.1034228	2.627	0.02752	*
FERTILIZERS.13	0.1676412	0.2614091	0.641	0.53732	
CHEMICALS.13	0.9299800	0.5972656	1.557	0.15388	
PHARMACEUTICALS.13	-0.2974409	0.4881507	-0.609	0.55738	
LEATHER.13	-0.2003960	0.1419825	-1.411	0.19175	
ENGINEERING.13	-0.4583712	0.1905157	-2.406	0.03951	*
PAPERBOARD.13	0.6674691	0.2375861	2.809	0.02040	*
QIM.14	-1.1321376	0.3328775	-3.401	0.00786	**
GDP.14	-0.4663835	0.9289439	-0.502	0.62768	
FDI.14	-0.0008915	0.0133701	-0.067	0.94829	
CPI.14	-2.6120678	3.5655003	-0.733	0.48245	
WPI.14	2.3626650	3.4053375	0.694	0.50531	
BOT.14	-0.2638477	0.2140374	-1.233	0.24891	
FBT.14	0.0654125	0.2455168	0.266	0.79591	
TEXTILE.14	1.3719808	0.9883179	1.388	0.19847	
COKEPETROLEUM.14	-0.2429591	0.2149302	-1.130	0.28753	
NONMETALLIC.14	-0.2864131	0.3538834	-0.809	0.43919	
IRONSTEEL.14	-0.2131796	0.2682933	-0.795	0.44731	
ELECTRONICS.14	-0.0203945	0.0753142	-0.271	0.79266	
RUBBER.14	1.1671084	0.7248394	1.610	0.14182	
WOOD.14	-0.0648907	0.0915547	-0.709	0.49642	
FERTILIZERS.14	-0.3979226	0.3765744	-1.057	0.31819	
CHEMICALS.14	-0.3322463	0.5232494	-0.635	0.54125	
PHARMACEUTICALS.14	0.1814621	0.4354478	0.417	0.68664	
LEATHER.14	0.0890923	0.1634927	0.545	0.59904	
ENGINEERING.14	0.1840353	0.1962101	0.938	0.37276	
PAPERBOARD.14	0.3433802	0.1632605	2.103	0.06477	.
QIM.15	0.6114587	0.3635397	1.682	0.12687	
GDP.15	1.1521781	1.0798111	1.067	0.31375	
FDI.15	0.0100527	0.0151123	0.665	0.52260	
CPI.15	-1.3806082	4.1506073	-0.333	0.74703	
WPI.15	0.3957193	4.2044629	0.094	0.92708	
BOT.15	0.8791177	0.3489203	2.520	0.03279	*
FBT.15	-0.0520314	0.2749402	-0.189	0.85410	
TEXTILE.15	0.5339788	0.6593234	0.810	0.43889	
COKEPETROLEUM.15	-0.4573785	0.2366029	-1.933	0.08525	.
NONMETALLIC.15	-0.0736314	0.3405364	-0.216	0.83364	
IRONSTEEL.15	0.0005775	0.1788491	0.003	0.99749	
ELECTRONICS.15	-0.1312506	0.0869104	-1.510	0.16528	
RUBBER.15	-0.6062232	0.4874570	-1.244	0.24505	
WOOD.15	-0.0628183	0.0815118	-0.771	0.46066	
FERTILIZERS.15	0.2101054	0.2529663	0.831	0.42770	
CHEMICALS.15	-1.1600515	0.4541607	-2.554	0.03098	*
PHARMACEUTICALS.15	0.4824279	0.3583359	1.346	0.21113	
LEATHER.15	0.0581257	0.1366733	0.425	0.68061	
ENGINEERING.15	0.5018393	0.2729304	1.839	0.09911	.
PAPERBOARD.15	0.3108085	0.2017833	1.540	0.15787	

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 9.468 on 9 degrees of freedom
Multiple R-squared: 0.9023, Adjusted R-squared: -0.1835
F-statistic: 0.831 on 100 and 9 DF, p-value: 0.6986

VAR Forecast



The study has shown that out of the 20 predictors/variables, 13, namely foreign direct investment, nonmetallic mineral products, food beverages and tobacco, balance of trade, chemicals, wood products, engineering products, paper and board, coke and petroleum, rubber products, gross domestic product, wholesale and consumer price indices show somewhat (Table 1) affiliation between themselves and the QIM. Rest of the variables (textile, pharmaceuticals, iron and steel, fertilizers, electronics and leather products) did not provide a suitable affiliation; hence we can say that these do not appear to be the pressure points for the LSM industry of Pakistan.

SVAR Estimation Result

Estimated A matrix:

	QIM	GDP	FDI	CPI	WPI	BOT	FBT	TEXTILE	COKEPETROLEUM	NONMETALLIC	IRONSTEEL	ELECTRONICS	RUBBER
QIM	1.00000	0.000000	0.0000000	0.00000	0.00000	0.00000	0.0000	0.0000	0.00000	0.00000	0.00000	0.00000	0.00000
GDP	0.06957	1.000000	0.0000000	0.00000	0.00000	0.00000	0.0000	0.0000	0.00000	0.00000	0.00000	0.00000	0.00000
FDI	8.08334	-2.956764	1.0000000	0.00000	0.00000	0.00000	0.0000	0.0000	0.00000	0.00000	0.00000	0.00000	0.00000
CPI	0.24736	0.075823	0.0005287	1.00000	0.00000	0.00000	0.0000	0.0000	0.00000	0.00000	0.00000	0.00000	0.00000
WPI	0.07173	0.029125	0.0016113	-0.69032	1.00000	0.00000	0.0000	0.0000	0.00000	0.00000	0.00000	0.00000	0.00000
BOT	0.30887	-0.113919	0.0059765	-0.33614	-0.46582	1.00000	0.0000	0.0000	0.00000	0.00000	0.00000	0.00000	0.00000
FBT	-0.32340	0.063342	-0.0064875	-0.04286	-0.51593	0.05552	1.0000	0.0000	0.00000	0.00000	0.00000	0.00000	0.00000
TEXTILE	-0.53274	-0.193988	-0.0129078	0.05345	-0.15956	-0.41115	-0.3510	1.0000	0.00000	0.00000	0.00000	0.00000	0.00000
COKEPETROLEUM	0.58298	0.285833	0.0287610	0.17533	0.07323	0.32172	-0.5440	-0.6161	1.00000	0.00000	0.00000	0.00000	0.00000
NONMETALLIC	0.42257	-0.054084	0.0003809	-0.26891	-0.46058	0.06665	-0.5013	-0.1737	-0.27984	1.00000	0.00000	0.00000	0.00000
IRONSTEEL	-0.61493	-0.004207	-0.0962997	0.40073	0.88304	-0.40753	0.9957	-3.9781	-0.64750	0.67033	1.00000	0.00000	0.00000
ELECTRONICS	0.44126	0.229399	0.2296538	-0.01311	-0.48848	0.39892	-0.8237	0.8801	2.29367	0.21845	-0.81431	1.00000	0.00000
RUBBER	-0.06843	-0.095091	0.0014829	-0.17181	-0.18692	0.19419	-0.3141	-0.5095	0.12377	-0.51358	0.15512	0.06260	1.00000
WOOD	-0.47208	-0.339611	-0.1157395	-0.51025	-0.41632	-1.92072	1.7489	-2.8064	-1.31972	0.23174	0.92522	-0.21137	0.16848
FERTILIZERS	-0.09537	-0.252718	-0.0488775	-0.02547	-0.01229	-0.71541	-0.4011	-0.6518	0.05359	-0.25198	0.06218	-0.15367	-0.50018
CHEMICALS	-0.07302	-0.098323	-0.0044418	-0.09966	-0.08508	-0.30504	-0.2665	-0.5108	-0.15372	-0.27401	0.05863	0.01202	-0.09372
PHARMACEUTICALS	-0.12465	-0.078897	-0.0056562	0.18718	0.19287	-0.22645	-0.5235	-0.1990	-0.32149	0.18051	0.05503	0.11808	0.14306
LEATHER	-0.79783	-0.133856	-0.0428218	-0.79233	-1.04242	-0.56321	-0.8325	0.2831	-0.02386	0.03475	-0.17805	-0.08272	0.20541
ENGINEERING	0.44614	0.051690	0.0146384	0.06149	-0.01187	0.97653	0.5445	-0.4479	-0.73257	-0.28794	0.16750	-0.20933	0.08476
PAPERBOARD	0.22590	0.020843	-0.0914130	1.24128	1.25067	-0.85343	0.3625	-2.4031	2.18928	0.03384	0.61636	0.24135	0.28051

	WOOD	FERTILIZERS	CHEMICALS	PHARMACEUTICALS	LEATHER	ENGINEERING	PAPERBOARD
QIM	0.00000	0.0000	0.00000	0.0000	0.0000	0.000	0
GDP	0.00000	0.0000	0.00000	0.0000	0.0000	0.000	0
FDI	0.00000	0.0000	0.00000	0.0000	0.0000	0.000	0
CPI	0.00000	0.0000	0.00000	0.0000	0.0000	0.000	0
WPI	0.00000	0.0000	0.00000	0.0000	0.0000	0.000	0
BOT	0.00000	0.0000	0.00000	0.0000	0.0000	0.000	0
FBT	0.00000	0.0000	0.00000	0.0000	0.0000	0.000	0
TEXTILE	0.00000	0.0000	0.00000	0.0000	0.0000	0.000	0
COKEPETROLEUM	0.00000	0.0000	0.00000	0.0000	0.0000	0.000	0
NONMETALLIC	0.00000	0.0000	0.00000	0.0000	0.0000	0.000	0
IRONSTEEL	0.00000	0.0000	0.00000	0.0000	0.0000	0.000	0
ELECTRONICS	0.00000	0.0000	0.00000	0.0000	0.0000	0.000	0
RUBBER	0.00000	0.0000	0.00000	0.0000	0.0000	0.000	0
WOOD	1.00000	0.0000	0.00000	0.0000	0.0000	0.000	0
FERTILIZERS	0.04976	1.0000	0.00000	0.0000	0.0000	0.000	0
CHEMICALS	0.10958	-0.1015	1.00000	0.0000	0.0000	0.000	0
PHARMACEUTICALS	-0.05677	-0.4368	-0.18437	1.0000	0.0000	0.000	0
LEATHER	0.04778	0.3301	-0.16905	-0.4721	1.0000	0.000	0
ENGINEERING	-0.20403	0.1620	0.09623	-0.1976	-0.7276	1.000	0
PAPERBOARD	0.49210	0.1606	-0.93843	-0.7943	-0.3112	-1.781	1

4. COMMENTS AND CONCLUSION

Based on the obtained results it can be advised to the policy makers of the country to focus on the production and improvement of these certain variables/industrial sectors that add up to boost the QIM and hence the overall economic growth of Pakistan, namely : foreign direct investment, nonmetallic mineral products, food beverages and tobacco, balance of trade, wood products, engineering products, paper and board, coke and petroleum, chemicals and rubber products. More comprehensive and deep study/analysis is required for better and clearer snapshot of this sector.

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FORECASTING AREA AND YIELD FOR MAIZE CROP OF PUNJAB, PAKISTAN FOR 2021-2030

Omer Bashier[§], **Ghulam Mustafa, Khadim Hussian**
Muhammad Shazad and **Abdul Waheed**
Crop Reporting Service, Agriculture Department, Vehari, Pakistan
[§]Email: advehari@hotmail.com

ABSTRACT

This study attempts to forecast area and yield for maize crop of Punjab by using Auto Regressive Integrated Moving Average (ARIMA) models. Using the time series data for the year 1948 to 2020, area and yield for maize crop were forecasted for 10 years starting from 2021 to 2030. ARIMA (0.1.0) and ARIMA (1.1.0) were found to be the best model for area and yield of maize crop respectively. Some diagnostics tests were also performed on fitted models and found well fitted. Forecasted area and yield will increase for the year 2021 to 2030.

INTRODUCTION

Agriculture is vital role to any country's overall growth, so it must be developed. For example, farmers contribute for roughly 19 percent of Pakistan's total GDP. This percentage alone indicates that Pakistan is a highly developed country. Because agriculture takes up such a large quantity of area, the country is reliant on it. Maize is one of the most widely grown crops. It is one of the world's most important cereal crops, and it helps to ensure food security in the majority of poor countries. It's cultivated in more than 70 nations throughout the world. The top maize-producing nations are:

The United States of America, China, Brazil, Mexico, Indonesia, India, France, and Argentina are among the countries involved. In Pakistan, maize is at third position by area, after wheat and rice. Maize is an important crop in Pakistan for human consumption, poultry feed, and livestock fodder, as well as a raw material for industry. Prussic acid, oxalic acid, and ergot toxicity are not present in maize fodder at any stage of plant development. Maize is the most suited crop for silage and is known as the "King of Silage Crops." It is cultivated on 0.974 million hectares for grain production, yielding 3.707 million tons of grain per year with an average yield of 3805 kg/ha.

Maize is primarily a rainfedkharif crop that is planted shortly before the monsoon arrives and harvested once the monsoon has passed. Despite this, Pakistan corn yields are much lower than those of other major corn-producing countries. Increased acreage under hybrids, adoption of stronger genetics, and enhanced agronomic practices all have the potential to boost Pakistan's maize production significantly. As maize crop is most important crop for cereal production. By keeping this importance in view there is need to forecast yield and cultivation area of maize crop in Punjab. Government needs correct facts and figures about area and production of major crops in advance for policy making.

Many studies have made significant progress in estimating the acreage and yield of wheat and rice crops, while maize forecasting has received only minor attention. Previously, a number of crop forecasting models were developed. A few of them are by Azhar et al. (1973), Amir & Akhtar (1984), Sher and Ahmad (2008) for wheat crop and Khan and Khan (1988), Maria and Tahir (2011) for rice crop, Maqsood et al. (2004) for sugarcane crop, Badmus and Ariyo (2011) and Tahir and Habib (2013) for maize crop.

Number of researchers has done work to forecast the area and production of different major crops like Wheat and Rice. Many researchers used ARIMA model for forecast area and Yield for different crops. Sharma et al. (2018) forecasted the area and production of maize crop in Nigeria by using the ARIMA model. Verma, S. (2018) used modelling and forecasting maize yield in India using ARIMA and State Space Models. But little work has done on forecasting of yield and production of maize crop in Punjab province of Pakistan. Our main focus in this paper is forecast the yield and area of maize crop in Punjab by using the ARIMA time Series model. Esther, N & Magdaline, N (2017) diagnosed by plotting ACF and PACF of the residual of best fit model, that was ARIMA (1,1,2) to check the autocorrelation in the error terms. They reveal that ARIMA (1,1,2) was the best model to forecast pulses (lentils) yield in Kenya. They predicted the lentil yield by using the ARIMA model in Kenya. Badmus & Ariyo (2011) forecasted the area and production of maize crop in the country Nigeria by using the ARIMA (1,1,1) and (2,1,2) models. Ramesh et al. (2014) found the most suitable ARIMA model for maize yields that was ARIMA (1,1,0).

METHODOLOGY

Modeling and forecasting of agriculture commodities have traditionally been carried out by using various econometric modeling techniques. The reason for using ARIMA model is that ARIMA is the most general class of forecasting models.

Data Analysis Strategy:

Data was collected by Crop Reporting Service for the period of 1948-2020. The software SPSS ver. 26 was used to analyze the time series data. ARIMA model by employing Box- Jenkins methodology was used for forecasting.

ARIMA:

The autoregressive integrated moving average [ARIMA] model is one of the most well-known and extensively utilized models. This model combines the autoregressive and moving average models into one. In nature, time series are frequently non-stationary. To get a stationary time series, we must use the differencing term d. to convert a non-stationary series to a stationary one. As a result, the ARIMA model is the generic version of the ARMA model (p,d,q).

In the ARIMA model, P represents the order of the autoregressive term, d represents the order of the differentiating term, and q represents the order of the moving average term.

$$\Delta^d z_t = (a_1 y_{t-1} + \cdots + a_m y_{t-m} + e_t + \beta_1 e_{t-1} + \beta_2 e_{t-2} + \cdots + \beta_n e_{t-n}).$$

Box-Jenkins Approach:

The Box-Jenkins methodology is extensively used for univariate time series modeling and is recognized as the most successful forecasting tool. It is utilized in planning and prediction. The Box-Jenkins methodology differs from others in that it selects the best predictive model based on the previous behavior of a variable. It is thought that any time series model may be represented by one of these three types of models. The Box-Jenkins approach is used to estimate and predict the univariate time-series model. Only stationary data may be used with this approach.

The steps for Box Jenkins are as follows:

- Identification of model
- Estimation of parameters
- Diagnostic checking
- Forecasting

Autoregressive models (AR): basis of forecasting is linear function of variable’s past values.

Moving Average models (MA): basis of forecasts is linear combination of past errors.

Autoregressive-Moving Average models (ARMA): combination of both categories.

RESULTS AND DISCUSSION

Identification of Time Series Model for Maize Crop:

The first step in identifying the perfect model is to find out the trend and stationery of data which can be assessed from the squence chart and correlogram .The sequence chart of maize area and yield (original series) as depicted in Figure 1 and Figure 2 respectively which are indicated that the area and yield of maize crop show the increasing trend over the years and thus indicated non–stationary of data. For making the stationary, there is need to taken the regular first differnce.

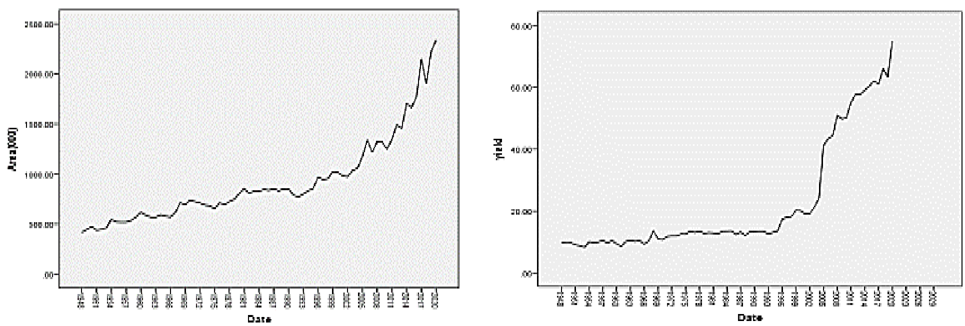

Figure 1: Line Diagram of Maize Area (000) and Yield in Punjab (Original Series)

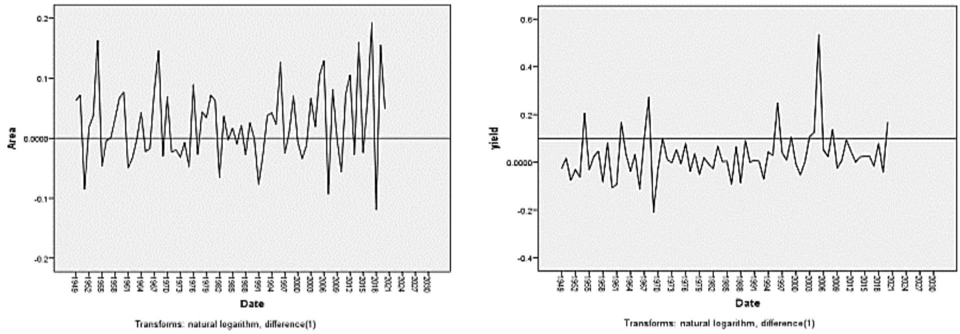

Figure 2: Line Diagram of Maize Area and Yield in Punjab after 1st Difference

After taking the first difference, Figure 3 and Figure 4 indicate after the first difference, the series become stationary for area and yield of maize crop in Punjab respectively. Thus, the value of d was fixed as 1. In order to identify the order of Autoregressive (AR) for the value ‘p’ and ‘the order of Moving Average ‘MA’ for the value of q. Correlograms of autocorrelation functions (ACF) and partial autocorrelation functions (PACF), respectively were examined. The correlogram of autocorrelation function (ACF) of differenced series (Figure 3) indicates that the auto correlation function falls immediately after one lag, hence the value of ‘q’ was decided to be ‘0’. Further, the correlogram of partial autocorrelation function (PACF) of differenced series (Figure 3) indicates that the auto correlation function falls immediately after one lag, hence the value of ‘p’ was decided to be ‘0’. Thus, the ARIMA (0,1,0) model may be selected for parameter estimation, model validation and forecasting of maize Areain Punjab. The correlogram of autocorrelation function (ACF) and partial autocorrelation function (PACF) of differenced series (Figure 3) indicates that the auto correlation function and partial autocorrelation function has one lag significant’. Thus, the ARIMA (1,1,0) model may be selected for parameter estimation, model validation and forecasting of maize Area in Punjab. The use of other diagnostics such as large value of R^2, minimum value of MAPE &Normalized BIC, significance of AR and MA parameters also confirms the selection of the ARIMA (0,1,0) and ARIMA (1,1,0) model for maize area and yield in Punjab respectively (Table 2).

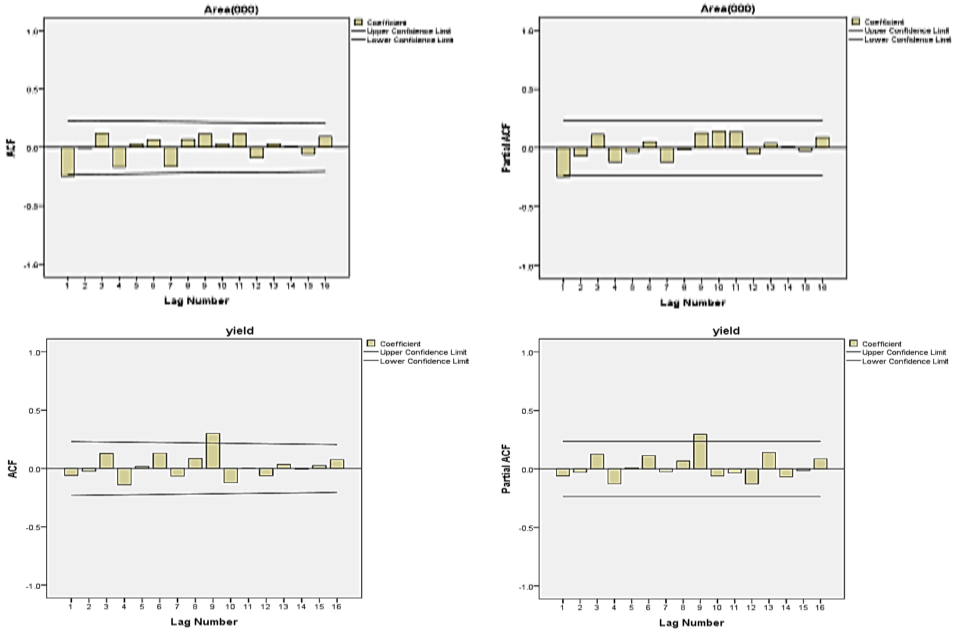

Figure 3: Correlograms of ACF and PACF of Differenced Series for maize Area and Yield in Punjab Pakistan

Table 1
ARIMA Models Fitted for Time Series Data of Maize Area and Yield and Corresponding Selection Criteria, i.e. R2, MAPE and Normalized BIC

Parameter	Model	R2	MAPE	Normalized BIC
Area	**(1,1,1)**	**0.977**	**4.756**	**8.649**
	(0,1,0)	0.977	4.833	8.498
	(0,1,1)	0.977	4.749	8.551
	(1,1,0)	0.978	4.758	8.547
Yield	**(1,1,0)**	**0.992**	**4.235**	**1.449**
	(1,1,1)	0.627	4.601	1.515
	(0,1,1)	0.991	4.625	1.465
	(0,1,0)	0.99	4.995	1.501

Parameter Estimates for Maize Area and Yield in Punjab:

After identifying the suitable ARIMA (0, 1, 0) and ARIMA (1, 1, 0) for area and yield of maize crop in Punjab, the parameters of identified model were assessed and are presented in Table 2.

Table 2
Estimates of ARIMA (2,1,0) Model for Maize Production in India

	Model		**Estimate**	**Standard Error**	**Sig.**
Area	ARIMA (0,1,0)	Constant	0.024***	0.007	0.001
		Difference	1	0	0
		MA lag 1			
Yield	ARIMA (1,1,0)	Constant	0.016***	0.005	0.002
		Difference	1	0	0
		AR lag 1	-0.435***	0.122	0.001

***significant at 1% level; **significant at 5% level.

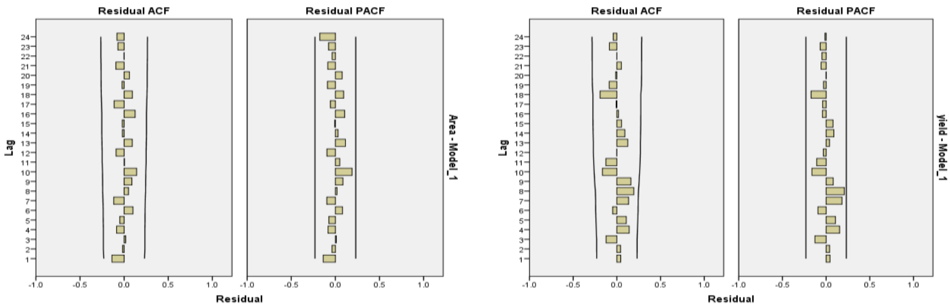

Figure 4: Residual Plots (Correlograms) of ACF and PACF for Identified ARIMA model (2, 1, 0)

Model Diagnostics:

The goodness of fit for the identified model was checked by plotting the residuals any systematic pattern, as shown in Figure 4. As the time series plots of ACF and PACF of the residuals of fitted ARIMA (0,1,0) model and ARIMA (1,1,0) model, exhibited a non-significant pattern, the given models were considered as valid for forecasting.

Forecast of Maize Area and Yield in Punjab:

After the model identification, estimation of its parameters and diagnostic checks, the forecasting of was made to know the future values of maize area and yield in Punjab by using the selected and fitted ARIMA (0,1,0) model and ARIMA (1,1,0) respectively. The forecasted values of maize area and yield in Punjab for the year 2021 to 2030 are presented in Table 4.

Table 4
Forecasting of Maize area and Yield with Confidence Intervals

	Years	Forecast	95% Lower Confidence Limit	95% Upper Confidence Limit
Area (000)	2021	2396.29	2122.69	2695.48
	2022	2458.73	2069.14	2900.78
	2023	2522.79	2040.30	3085.99
	2024	2588.53	2023.89	3263.53
	2025	2655.97	2015.27	3438.15
	2026	2725.18	2012.12	3612.34
	2027	2796.18	2013.10	3787.58
	2028	2869.04	2017.35	3964.88
	2029	2943.79	2024.29	4144.98
	2030	3020.50	2033.51	4328.45
Yield	2021	71.38	63.26	80.27
	2022	74.71	65.01	85.46
	2023	75.07	63.53	88.12
	2024	76.78	63.73	91.73
	2025	77.93	63.44	94.76
	2026	79.36	63.51	98.00
	2027	80.69	63.53	101.11
	2028	82.11	63.67	104.27
	2029	83.52	63.84	107.43
	2030	84.97	64.06	110.81

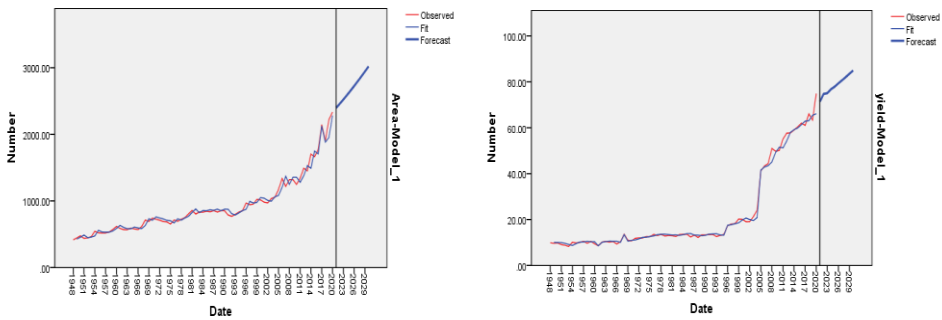

Figure 5: Predicted, Actual & Forecasted Values of Maize Area & Yield in Punjab

CONCLUSION

The Autoregressive integrated moving average (ARIMA) model is considered to be one of the best models when the data consists if at least 50 observations. The present study attempts at modeling and forecasting of maize area and yield in Punjab was done using Autoregressive integrated moving average (ARIMA) model. Autocorrelation function (ACF) and partial autocorrelation function (PACF) functions were estimated, which led to the identification and construction of ARIMA model. The best ARIMA

(0,1,0) model and ARIMA (1,1,0) model for maize area and yield in Punjab the fitted model indicated an increase in maize area in the next ten years from 23,96,290 in the year 2021 to 30,20500 in the year 2030 and similarly maize yield in the next ten years from 71.38 in the year 2021 to 84.97 in the year 2030.

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COMPARATIVE STUDY OF THE QUALITY OF URDU POETRY

Muhammad Ali and **Ammara Nawaz Cheema**
Department of Mathematics, Air University, Islamabad, Pakistan.
Email: ammara.au@gmail.com; ammara.cheema@mail.au.edu.pk

ABSTRACT

Poetry is written to share ideas, express emotions, and create imagery. Poetry remains an important part of art and culture. Urdu poetry is a rich tradition of poetry and has many different forms. Our purpose of the study is to be able to educate people about URDU POETRY. The main objective of the research is to identify the people who love the URDU POETRY and know the worth of the knowledge of it. We have collected data from different persons through questionnaire. Different statistical techniques including descriptive, graphs, F-test and regression model are accustomed for the comparative study of the quality of URDU POETRY. This study is useful to determine the role of URDU POETRY in the society.

1. INTRODUCTION

Poetry is a concentrated imaginative awareness of experience or a specific emotional response through language chosen and arranged for its meaning, sound, and rhythm. Poetry is the other way of using language. Like other forms of literature, poetry is written to share ideas, express emotions, and create imagery. Poets choose words for their meaning and acoustics, arranging them to create a tempo known as the meter. Some poems incorporate rhyme schemes, with two or more lines that end in like-sounding words. Now a day, poetry remains an important part of art and culture. Urdu poetry is a rich tradition of poetry and has many different forms. Today, it is an important part of the cultures of South Asia. According to Naseer Turabi there are five major poets of Urdu which are Mir Taqi Mir, Mirza Ghalib, Mir Anees, Allama Iqbal and Josh Malihabadi. Urdu poetry of the Indian subcontinent as we know it today did not take its final shape until the 17th century when it was declared the official language of the court. The 18th century saw a phenomenal rise in Urdu poetry when Urdu replaced Persian as the lingua franca of the region. Urdu poetry, as it is derived from Persian, Turkish and Arabic, acquired many conventions in its poetry that came from these languages. Urdu poetry is based on a system of measure. It is a quantitative expression and its form is very rigid. Many poets have specialized in the specific art of writing. Most have attempted ghazal, the most popular form and those whose fame reached the greatest heights have been poets of ghazal. Since each verse of a ghazal is an independent segment and a complete description of the topic (though there may be a chain of verses with the same theme), it requires a great deal of ability to express in the fewest words the most complex emotions.

Arabic, Persian, Sanskrit and English have combined to promote Urdu [1]. African poetry presents different aspects of love. It present love as an important emotion of mankind [2]. The great poets has acknowledged Mir's poetic talents. Mir's intellectual and artistic poetry is again unparalleled [3]. In Haider Ali Atash's poetry, the feelings of beauty and love and the themes of dervishes and ethics are beautifully seen [4]. Momin Khan Momin is a romantic poet. His poetry has a lot of inner quality [5]. Hasrat Mohani's poetry has its own example. He is called Raees-ul-Mutaghzalin [6]. In Faiz's poetry, along with the color of love, the social color is beautifully seen [7]. Ahmad Nadeem Qasmi wrote poetry in both ghazal and poem. His poetry reflects the political and social eras [8]. Criticism is present everywhere but people started criticizing our legendary poems [9]. Ghalib is a great poet. Ghalib's poetry has its own example. Dawood Kamal (1935-1987), who was a poet of English language, translated Ghalib's poetry into English [10]. The most famous Urdu language drama serial was Anarkali. In this play, Urdu language is beautifully described [11]. Kaleem Khariji is a prominent figure of short stories in contemporary Urdu literature. He does not use symbols, but his characters are capable of symbolical depth. The themes and subject matter of the stories in his book, Ghatiya Aadmi are derived from the social and political circumstances of Khyber Pakhtunkhwa [12]. Sir Syed stands the first Urdu essays writer. After the issuance of Tehzeeb ul Akhlaq in the last decade of the nineteenth century, women started writing essays to help bring about reforms and change the social status of women [13]. Khawar Ahmed’s uniqueness however, is that he discusses new aspects of a commonplace theme like ‘love’ [14]. The poet-philosopher Allama Muhammad Iqbal is one of those great scholars who have left a legacy behind to be followed by other scholars particularly in the area of how to deal with the West. His own reconciliatory approach in dealing between the West and the Islamic world should be an interesting one [15].

2. METHODOLOGY

In methodology, all concepts and tools that have been used in drawing the inference about comparative analysis of quality of URDU POETRY.

The main focus of this proposed study is in the statistical analysis of URDU POETRY. To illustrate the proposed methodology, we have taken the data from different people through Google form questionnaire. We have applied basic statistical analysis; including descriptive, graphs, F-test and regression model to draw the inference about URDU POETRY.

The fundamental main objective of this research is to consciousness of URDU POETRY among people and to provide essential knowledge about URDU POETRY. We have to distinguish poetry from other writings like Essays, Novels, etc.To understand hidden message of the poetry. To predict the rhyming lines increase the quality of poetry. To understand the ideology of the poem. To distinguishes vulgarity and romanticism in poetry. To understand figurative language in the poetry. To differentiate Iqbal's poetry with now a day’s poetry. We have taken the data from different people and cross checked the data.

3. RESULT AND DISCUSSION

3.1 Basic Statistical Analysis

Descriptive analysis of academic dismissal data is figured below.

Table 1

	Descriptive						
	Mean	Median	Mode	Variance	Skewness	Kurtosis	Range
Occupation	1.82	2	2	0.15	-1.685	0.858	1
Age	1.84	2	2	0.351	0.688	2.709	3
Gender	1.57	2	2	0.247	-0.306	-1.948	1
Poetry Meaning	1.09	1	1	0.143	4.569	20.242	2
Everyone can write poetry	2.07	2	2	0.263	0.61	2.521	3
Comparison with other genres of writing	1.47	1	1	0.531	1.39	0.999	3
Contribution of social media	1.54	1	1	0.702	1.038	-0.759	2
Relation	1.31	1	1	0.431	2.144	3.9	3
Role of rhyming lines	1.27	1	1	0.477	2.597	5.822	3

Table 1 shows that 70% of our questionnaire was filled by youngsters. It shows that mostly people says that poetry is the expression of feelings. It shows that everyone cannot write poetry because it has some rules and regulations. It is totally different from other genres of writing. It shows that mostly people say that internet and social media contributes the well being of URDU POETRY. Mostly people say that if rhyming lines are added in the poetry then the beauty of the poem is increased.

Table 2

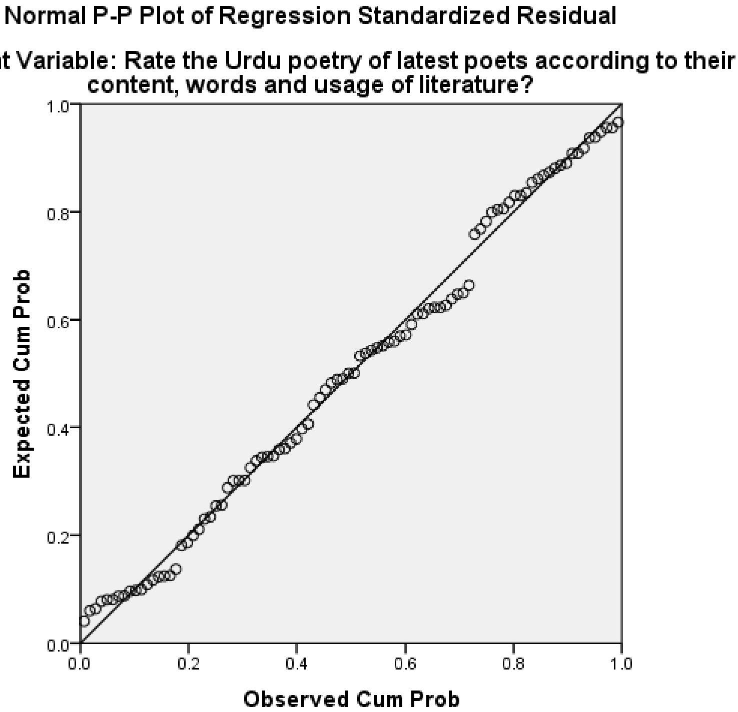
	Descriptive						
	Mean	Median	Mode	Variance	Skewness	Kurtosis	Range
Latest Urdu Poetry	1.91	2	1	0.896	0.406	-1.284	3
Limitations of Latest Urdu Poetry	1.85	2	1	0.773	0.395	-1.355	3
Scientific Way	1.79	2	2	0.707	1.31	1.68	3
Preference	1.91	2	2	0.595	1.298	2.348	3
Hidden Message in Poetry	1.12	1	1	0.169	4.64	26.618	3
Romantic Poetry or Serious Poetry	2	2	2	1.097	1.664	2.793	4
Iqbal’s Poetry	1.39	1	1	0.843	2.534	6.197	4
Kinds of Poetry	1.67	2	1	0.546	1.106	1.376	3
Where you Read Poetry	2.23	2	2	1.45	0.781	-0.529	4
Rate the Urdu Poetry	3.07	3	3	1.79	-0.029	-0.883	4

Table 2 shows that almost 45% of people believes that the latest poetry destroy our new generation. But 30% of people say that it depends on the context of poetry and also the mentality of the person. It shows that most of our latest poetry is only limited to romanticism and vulgharity. 60% of people believes that there is not any scientific way to determining the mind of poet. It shows that our new generation prefers simple romantic poetry instead of pure URDU POETRY because of movies and dramas. 80% of people say that our youth do not focus on the hidden message of poetry because of lack of understanding of Urdu literature. It shows that the preference of people about poetry varies according to their nature. Mostly people says that poetry is only limited for lyrics of music, videos or statuses. Mostly people use poetry from internet without searching on it.

3.2 Basic Regression Model Analysis

In this section we test the basic regression model for given data before drawing further conclusions based on the model.

Model	Unstandardized Coefficients		Standardized Coefficients	t	Sig.
	B	Std. Error	Beta		
(Constant)	1.892	1.166		1.622	.109
X_1	.061	.376	.017	.163	.871
X_2	.165	.291	.063	.566	.573
X_3	-.065	.201	-.036	-.325	.746
X_4	.086	.196	.054	.439	.662
X_5	-.030	.225	-.015	-.134	.894
X_6	.367	.225	.190	1.629	.107
X_7	.350	.185	.248	1.897	.061
X_8	.219	.189	.144	1.155	.252
X_9	-.090	.202	-.057	-.447	.656
X_{10}	-.211	.196	-.121	-1.075	.286
X_{11}	.154	.356	.047	.432	.667
X_{12}	-.143	.166	-.098	-.861	.392
X_{13}	-.014	.199	-.008	-.070	.944
X_{14}	-.031	.122	-.028	-.253	.801



3.3 F-test

The F-test is designed to test if two population variances are equal. It does this by comparing the ratio of two variances. So, if the variances are equal, the ratio of the variances will be 1.The result is always a positive number .The equation for comparing two variances with the f-test is:

$$F = \frac{S_1^2}{S_2^2}.$$

Rate the Urdu poetry of latest poets according to their content, words and usage of literature?

	Sum of Squares	df	Mean Square	F	Sig.
Between Groups	1.760	2	.880	.486	.617
Within Groups	164.719	91	1.810		
Total	166.479	93			

4. CONCLUSIONS

In this article, the comparative study of the quality of URDU POETRY is examined. The extensive statistical analysis is worked to assess and exhibit a few noteworthy highlights about the URDU POETRY under the real data study. The findings conclude that many people like to study URDU literature but many other people don't know any brief description about URDU literature. Because of lack of URDU literature's knowledge, they are unaware of the significance of URDU POETRY. Our research helps us to find those people who love URDU POETRY and know the worth of it.

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JOIN-POINT REGRESSION ANALYSIS TECHNIQUE FOR ASSESSING TIME TRENDS IN BREAST CANCER INCIDENCE RATES IN KARACHI, PAKISTAN

Sidra Zaheer[1,2] and **Farah Yasmeen**[1]

[1] Department of Statistics, University of Karachi, Karachi, Pakistan. Email: sidra.zaheer@duhs.edu.pk

[2] School of Public Health, Dow University of Health Sciences, Karachi, Pakistan. Email: riazfarah@yahoo.com

ABSTRACT

In Pakistan, one out of every nine women is at risk to be inflicted with breast cancer and the age-standardized incidence rates of breast cancer is one of the highest in Pakistan among other Asian countries. The aim of this study was to apply the Join-point Regression Analysis (JRA) technique for assessing time trends in breast cancer incidence in Karachi, Pakistan.

Observed age-specific breast cancer incidence cases for females from 2004 to 2015 were used to model join-point regression for each age group, 15-19, 20-24, 25-29, 30-34, 35-39, 40-44, 45-49, 50-54, 55-59, 60-64, 65-69, 70-74, 75+. Age-specific women breast cancer incidence rates were calculated as the number of new cases of breast cancer divided by the corresponding population at risk during a year, expressed as the rates per 100,000 persons per year. The trends in age-specific breast cancer incidence rates from 2004 to 2015 were explored using Join-point Regression Analysis by fitting a series of joined straight lines on a log scale, and average annual percentage changes (aAPC) were also calculated to assess the proportion with which the increase or decrease in incidence has occurred.

Trend in breast cancer incidence rates increased significantly from 2004 to 2015 by an average APC of 4.6% (95% CI = 2.0%, 7.3%, p-value 0.002). One join-point (year 2011) was identified as a year of change of trend and fastest increase was found in incidence in the period 2011-2015 (aAPC: 14.0% (95% CI = 5.7%, 23.0%, p-value 0.008). Fastest significant increase in incidence was observed during 2011-2015 in the age group 40-44, 55-59 and 75+ years old. One and two join-points regression models identified a sustained increase in breast cancer incidence among older and post-menopausal women in Pakistan.

1. INTRODUCTION

Joinpoint regression or trend analysis in cancer epidemiology is used to determine change in disease incidence over time. Joint point regression is also used to assess at what speed the increase or decrease has occurred in disease incidence. (Jiang, Qiu, and Hatcher 2010, Qiu et al. 2009, Clegg et al. 2009, Saad et al. 2018). Joinpoint models developed for cancer incidence can provide useful information in epidemiological studies to uncover changes in time trend of mortality and incidence of disease. Thus, to evaluate breast cancer incidence trends, researchers can estimate the joinpoints in different years and also can find when the incidence started rising, reached its peak, at what speed and type of changes occurred. It can even determines the point to which incidence has returned to the background trend before the change occurred. (Kim et al. 2000). The information obtained can be very helpful in assessment and planning for community's health plans as well as for national planning.

In the current study, we assessed the trends of age-specific incidence in breast cancer by using joinpoint regression analysis, and identified whether age-specific breast cancer incidence has changed or not, over the time. Average annual percentage changes (aAPC) were also calculated for each age group to assess the proportion with which the increase or decrease has occurred.

2. METHODOLOGY

2.1 Joinpoint Regression Model

The Joinpoint regression model is very useful in describing changes in trend data. The model is composed of few of continuous linear phases. Line segments are joined at points called change points or joinpoints. It identifies statistically significant trend change points (joinpoints) and the Annual Percent Change (APC) in each trend segment using a Monte Carlo permutation method (Autier et al. 2011, Kim et al. 2000 and Rea et al. 2017).

The APC is easy to calculate and interpret. If the percentage change in the rates is constant every year, the incident trend is linear on the log scale. Thus, for this particular reason, to estimate the APC for a series of data, the following regression model is used:

$$log(\,E[y_{it}]) = \alpha_i + \beta_i \times t + \varepsilon \tag{1}$$

where log ($\widehat{\mathrm{E}}[\mathrm{y}_{\mathrm{it}}]$) is the natural log of the rate in year *T: t=1,2,3,…, n.*

If we denote slope coefficients $\beta_i s$ as the slope coefficients for each segment in the preferred range of years, and the $w_i s$ as the length of each segment in the range of years, then we can write APC with estimated parameters as

$$APC_i = \left(e^{\widehat{\beta_i}} - 1\right) \times 100. \tag{2}$$

The average APC (aAPC) over any fixed interval is computed by the approach of weighted average of the slope coefficients of the underlying join-point regression line, where the weights are taken equal to the length of each segment over the interval (Dragomirescu et al. 2019). The average APC can be calculated first on the log scale and in the final step are then transformed back to the annual percent change.

$$aAPC = \left\{e\left(\frac{\sum w_i \beta_i}{\sum w_i}\right) - 1\right\} \times 100. \tag{3}$$

2.2 Fitting Joinpoint Model using Joinpoint Regression Program Software

For the analysis of trends using joinpoint models, “Joinpoint Regression Program” software was used. The Joinpoint Regression Program is available on website for public use, developed by National Cancer Institute (NCI) (Program Version 4.5.0.1 - June 2017). This software is used by many registries around the world for cancer rates. Cancer trends reported in NCI publications are calculated using this program to analyze rates calculated by the SEER*Stat software.

2.3 Age-Specific Incidence Rates of Breast Cancer: 2004-2015

Observed age-specific breast cancer incidence cases for females from 2004 to 2015 were used to model join-point regression for each age group, 15-19, 20-24, 25-29, 30-34, 35-39, 40-44, 45-49, 50-54, 55-59, 60-64, 65-69, 70-74, 75+. Age-specific women breast cancer incidence rates were calculated as the number of new cases of breast cancer divided by the corresponding population at risk during a year, expressed as the rates per 100,000 persons per year.

2.4 Data Analysis

All statistical analysis was performed using Joinpoint Regression Program to model age-specific incidence rates in female for the period 2004-2015. First, average annual percent change (aAPC) in breast cancer rates was estimated for the crude age-specific breast cancer incidence for overall trend and then computed for zero joinpoint to 3 joinpoints models (Trend 1, Trend 2, Trend 3). Results are shown in tabular and graphical forms. Further, average annual percent changes (aAPCs) in age-specific breast cancer incidence were also calculated using joinpoint regression with zero joinpoint, 1 joinpoint and 2 joinpoints models.

3. RESULTS

Figure 1 showed that overall age-specific breast cancer incidence rates increased from 2004 to 2015 by an annual APC of 4.6% (95% CI = 2.0%, 7.3%, p-value 0.002). Result shows a significant increase in crude breast cancer incidence among women. There is no significant trend was found for 1 joinpoint model (trend 2004-2011), while trend 2011-2015 (aAPC: 12.4% (95% CI = 1.3%, 24.8%, p-value 0.032), and for 2 joinpoint models, trend 2011-2015 (aAPC: 14.0% (95% CI = 5.7%, 23.0%, p-value 0.008) were found considerably significant high percentage change.

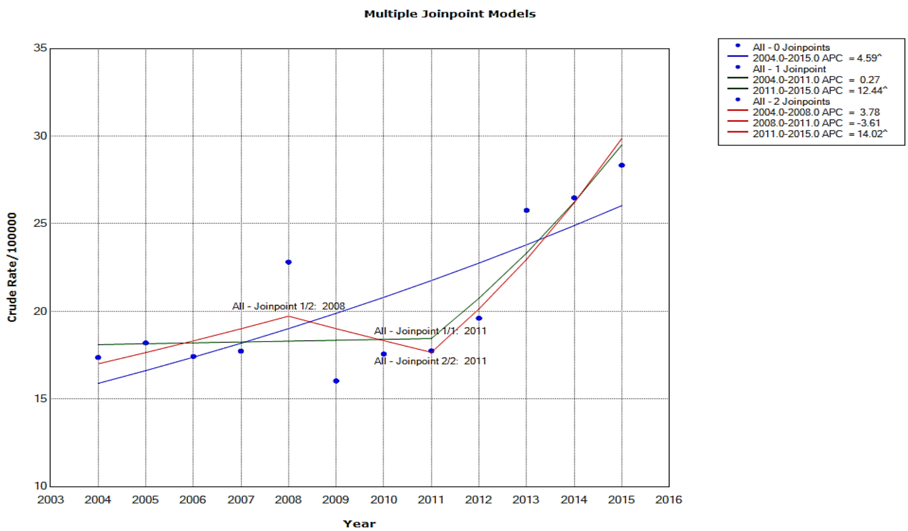

Figure 1: Crude Breast Cancer Incidence Rates/100000 Women. ^APC (Annual Percentage Change) Represents Significant Difference from Zero at Alpha=0.05

3.1 Age-Specific Average Annual Percent Change (aAPC)-Zero Joinpoint Model

Average APCs in age-specific breast cancer incidence rates for overall trend 2004-2015 are shown in Table 1. There is a significant linear change for during period 2004-2015, a significant increase in the breast cancer incidence is expected from 2004 to 2015 among women aged 75 and above (aAPC = 9.3%, 95% CI 3.3%, 15.6%, p-value 0.005), women aged 40-44 years (aAPC = 6.5%, 95% CI = 3.1%, 10.0%, p-value 0.001) , and the highest decreasing annual APC of -11.4% (95% CI = -17.3%, -5.0%, p-value 0.003) was observed among women aged 15-19 years old.

Table 1
Average Annual Percent Change (aAPC) in Age-Specific Breast Cancer Incidence Rates. Overall (2004-2015)

Zero Joinpoint Model:				
Age-group	**Period**	**aAPC**	**95% Confidence Intervals**	**p-value**
15 – 19	2004-2015	-11.4	-17.3, -5.0	0.003*
20 – 24	2004-2015	3.9	-2.0, 10.2	0.178
25 – 29	2004-2015	3.2	-1.2, 7.7	0.137
30 – 34	2004-2015	4.8	0.7, 9.1	0.025*
35 – 39	2004-2015	3.2	-0.6, 7.2	0.092
40 – 44	2004-2015	6.5	3.1, 10.0	0.001*
45 – 49	2004-2015	5.8	2.8, 8.9	0.001*
50 – 54	2004-2015	5.0	2.2, 8.0	0.002*
55 – 59	2004-2015	6.1	2.1, 10.4	0.007*
60 – 64	2004-2015	5.2	3.3, 7.1	0.001*
65 – 69	2004-2015	3.4	0.3, 6.5	0.032*
70 – 74	2004-2015	5.8	2.7, 9.1	0.002*
75 & above	2004-2015	9.3	3.3, 15.6	0.005*

aAPC: average annual percentage change
*p-value is significant at 95% confidence intervals

3.2 Age-Specific Average Annual Percent Change (aAPC)-One Joinpoint Model

A significant increase in the breast cancer incidence from 2004 to 2013 is expected among women aged 60–64 years (aAPC = 3.6%, 95% CI 0.6%, 6.8%, p-value 0.025) and among women aged 40-44 (aAPC = 17.6%, 95% CI = 3.9%, 38.0%, p-value 0.017), 45-49 years (aAPC = 14.8%, 95% CI = 2.3%, 28.9%, p-value 0.025) and 50-54 years aged women (aAPC = 12.9%, 95% CI = 0.3%, 27.0%, p-value 0.045) for the trend period 2011-2015. (See details in Table 2)

3.3 Age-Specific Average Annual Percent Change (aAPC)-Two Joinpoints Model

Similarly, two joinpoints model showed that age-specific breast cancer incidence rates increased significantly from trend 1 (2004 to 2008) among women aged 60-64 by an annual APC of 8.8% (95% CI = 3.6%, 14.4%, p-value 0.009). There is no significant annual average percentage change was found in trend 2, however, aAPC of 18.9% (95% CI = 7.5%, 31.5%, p-value 0.009) was observed among women aged 40-44 for trend 3 period 2011-2015. The average APC increased significantly for women aged 40-69 years for period 2011-2015, details are described in Table 3.

Table 2
Average Annual Percent Change (aAPC) in Age-Specific Breast Cancer Incidence Rates for each Trend. (2004-2015)

One Joinpoint Model:								
Age-Group	**Trend 1**				**Trend 2**			
	Period	**aAPC**	**95% CI**	**p-value**	**Period**	**aAPC**	**95% CI**	**p-value**
15-19	2004-2015	-	-	-	-	-	-	-
20-24	2004-2010	-4.5	-23.6, 19.5	0.645	2010-2015	13.7	-9.1, 42.2	0.217
25-29	2004-2010	-0.01	-16.2, 19.3	0.998	2010-2015	6.6	-11.6, 28.8	0.444
30-34	2004-2010	0.15	-13.6, 16.2	0.981	2010-2015	10	-6.3, 29.2	0.205
35-39	2004-2010	-4.2	-14.5, 7.3	0.398	2010-2015	12.2	-1.4, 27.9	0.074
40-44	2004-2011	0.43	-6.1, 7.4	0.885	2011-2015	17.6	3.9, 33.0	0.017*
45-49	2004-2011	0.88	-5.2, 7.4	0.752	2011-2015	14.8	2.3, 28.9	0.025*
50-54	2004-2011	0.78	-5.3, 7.3	0.779	2011-2015	12.9	0.3, 27.0	0.045*
55-59	2004-2011	1.5	-8.7, 13.0	0.742	2011-2015	13.9	-4.4, 35.8	0.123
60-64	2004-2013	3.6	0.6, 6.8	0.025*	2013-2015	15.4	-10.2, 48.5	0.220
65-69	2004-2011	-2.3	-6.9, 2.5	0.293	2011-2015	14.1	4.1, 25.2	0.011*
70-74	2004-2006	-1.9	-48.2, 85.7	0.945	2006-2015	6.7	1.8, 11.8	0.013*
75 & above	2004-2007	39	-22.8, 150.7	0.227	2007-2015	5.8	-2.4, 14.9	0.145

aAPC: average annual percentage change
CI: confidence intervals, *p-value is significant at 95% confidence intervals

Table 3
Average Annual Percent Change (aAPC) in Age-Specific Breast Cancer Incidence Rates for each Trend (2004-2015)

Two Joinpoints Model:												
Age-group	Trend 1				Trend 2				Trend 3			
	Period	aAPC	95% CI	p-value	Period	aAPC	95% CI	p-value	Period	aAPC	95% CI	p-value
15-19	2004-2015	-	-	-	-	-	-	-	-	-	-	-
20-24	2004-2010	-6.9	-29.6, 23.2	0.519	2010-2013	25.5	-70.3, 429.6	0.683	2013-2015	-4.2	-68.0,86.3	0.918
25-29	2004-2010	-1.7	-22.1, 23.9	0.843	2010-2013	14.6	-64.6, 271.3	0.764	2013-2015	-6.8	-65.2, 149.9	0.852
30-34	2004-2010	-2.2	-14.1, 11.3	0.659	2010-2013	20.9	-39.8, 142.9	0.492	2013-2015	-7.8	-47.1, 60.6	0.702
35-39	2004-2010	-4.8	-19.1, 12.0	0.449	2010-2013	15.2	-54.4, 191.5	0.693	2013-2015	7.2	-48.5, 123.4	0.804
40-44	2004-2008	3.2	-9.7, 17.9	0.552	2008-2011	-2.7	-35.9, 47.6	0.864	2011-2015	18.9	7.5, 31.5	0.009*
45-49	2004-2008	2.9	-14.9, 24.4	0.698	2008-2011	-1.3	-44.2, 74.5	0.949	2011- 2015	15.7	0.3, 33.6	0.047*
50-54	2004-2008	5.8	-11.3, 26.3	0.424	2008-2011	-4.7	-44.9, 64.8	0.817	2011-2015	15.2	0.04, 32.7	0.050*
55-59	2004-2008	12.5	-12.0, 44.0	0.254	2008-2011	-8.9	-53.0, 76.5	0.714	2011-2015	18.5	1.4, 38.4	0.039*
60-64	2004-2008	8.8	3.6, 14.4	0.009*	2008-2012	-0.8	-7.3, 6.1	0.752	2012-2015	14.1	7.6, 21.0	0.003*
65-69	2004-2008	2.9	-11.1, 19.3	0.608	2008-2011	-8.3	-41.1, 42.7	0.615	2011-2015	16.9	4.1, 31.2	0.020*
70-74	2004-2009	1.4	-14.8, 20.9	0.829	2009-2012	14.4	-42.9, 129.5	0.620	2012-2015	0.63	-25.3, 35.6	0.956
75 & above	2004-2010	21.7	-6.5, 58.5	0.108	2010-2013	-5.1	-65.0, 157.8	0.891	2013-2015	14.7	-54.4, 189.2	0.701

aAPC: average annual percentage change
CI: confidence intervals, *p-value is significant at 95% confidence intervals

4. COMMENTS AND CONCLUSION

Joinpoint models developed to uncover changes in time trend of mortality and incidence of disease. Thus, to evaluate breast cancer incidence trends, we applied the Joinpoint regression analysis to estimate the joinpoints in different years and to find when the incidence started rising, reached its peak, at what speed and type of changes occurred. We used "Joinpoint Regression Program" software to fit joinpoint models, this software developed by National Cancer Institute (NCI).

Results showed that a significant increase in crude breast cancer incidence was cleared among women, likewise, overall age-specific breast cancer incidence rates increased from 2004 to 2015 by an annual APC of 4.6% (95% CI = 2.0%, 7.3%, p-value 0.002). There is no significant trend was found for the trend 2004-2011, while trend 2011-2015 had considerably significant high percentage change. For age-specific trend analysis, it has been observed that there is a significant increase in breast cancer incidence from 2004 to 2015 among women; aged 75 and above and women aged 40-44 years.

One joinpoint model revealed that a significant increase from 2004 to 2013 among women; aged 60–64 and women aged 40-49 years, whereas 50-54 years aged women had a significant increase for the trend period 2011-2015. Similarly, two joinpoints model showed that age-specific breast cancer incidence rates increased significantly from 2004 to 2008 (trend 1) among women aged 60-64 and significant increase was observed among women aged 40-44 for trend 3, period 2011-2015. The annual APC increased significantly for women aged 40-69 years for period 2011-2015.

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MODELING AND FORECASTING OF MONTHLY AVERAGE MAXIMUM SURFACE AIR TEMPERATE IN MULTAN USING SEASONAL ARIMA

Muhammad Abdul Hafeez [1], **Omer Bashier**[2], **Ghulam Mustafa**[2], **Khadim Hussian**[2], **Muhammad Shazad**[2] and **Abdul Waheed**[2]

[1] Pakistan Meteorological Department, Multan, Pakistan
[2] Crop Reporting Service, Agriculture Department, Vehari, Pakistan

ABSTRACT

The prime objective of this study is to forecast the Monthly Average Maximum Surface Air Temperature in Multan for 5 years from January-2022 to December-2026. This study will help the policy makers for future planning especially in the field of agriculture. For the purpose of analysis of this study R-software is used. For this study the data is taken from Pakistan Meteorological Department for the period of 10 years from January-2012 to December 2021. The data from January-2012 to December-2019 is used as a training data set and data from January-2020 to December-2021 is used as a test data to check the performance of the fitted model. In this study the Box-Jenkins methodology is used and best Seasonal Autoregressive Integrated Moving Average model is developed. On the basis of AIC and BIC, we have found two competing SARIMA models. SARIMA$(1,0,0)(1,1,1)_{12}$ and SARIMA$(1,0,1)(1,1,1)_{12}$ are selected suitable models for Monthly Average Maximum Surface Air Temperature and then on the basis of forecasting accuracy measures i.e. RMSE, MAE and MAPE SARIMA(1,0,0)(1,1,1)12 is selected as a best forecasting model. The forecasted values are fellow the pattern of the past values.

1. INTRODUCTION

Time series analysis and forecasting is one of the major tool used by scientists in meteorology and environmental field to study meteorological phenomenon like Rainfall, temperature and humidity. In this study a most popular time series technique “The Box-Jenkins methodology” is used in order to build an appropriate model. Because temperature is seasonal phenomenon with twelve months period therefore a seasonal autoregressive integrated moving average (SARIMA) model is used.

Significant changes in climate are taking place worldwide, the major cause in climate change is rise in temperature .Temperature is the most significant climatic factors; therefore it is important to understand its nature. Because in the fields of agriculture and business, temperature has a direct impact. So, this study will help the policy makers for better future planning in the field of Agriculture and Business. Excess in temperatures may also influence both human and livestock. As a result, it's essential to know how rapidly the temperature will rise.

Climate change is now one of the most serious environmental threats facing the entire world. It is one of the most serious threats to water resources, livelihoods, and forests. The temperature is a common meteorological variable that indicates how hot or cold it is. It influences not only plant and animal growth and reproduction, but also nearly all other meteorological variables such as evaporation rate, relative humidity, wind speed and direction

Several studies on meteorological phenomenon like rainfall, Minimum temperature, Maximum temperatures , Relative humidity ,Dew point etc. have already be done and in most of the study Time Series Analysis and Forecasting tools have been used in order to built best forecasting models. Nury, Hasan and Alam (2013): The primary goal of this study was to develop ARIMA models for short-term prediction of minimum and maximum temperatures for the years 2010 and 2011 for two temperature stations in the Sylhet division using the Box-Jenkins Methodology. Two meteorological variables Minimum and Maximum temperatures were use in this study. ARIMA (1,1,1)(1,1,1)12, ARIMA (1,1,1)(0,1,1)12 and ARIMA(1,1,1)(1,1,1)12, ARIMA (0,1,1)(1,1,1)12 were develop for Sylhet division and for Moulvibazar respectively. Burney, Barakzai and James (2017): aims of this research was to identify a suitable model for the forecasting of maximum temperature over a period of 12 months for Karachi city by using Box-Jenkins Methodology. One Meteorological variable “Maximum Temperature” was use in this study. On the basis of Likelihood and AIC model SARIMA (0,0,2) (2,1,1)12 finalized to forecast the future maximum monthly temperature of Karachi city of Pakistan. Goswami,

DATA AND METHODOLOGY

In this study two time series variables, Minimum temperature and Maximum temperature are taken. The minimum heating is the lowest temperature recorded within a 24-hour. And the maximum temperature is the highest temperature recorded within a 24-hour. Data is collected for 10 years from January-2012 to December 2021 from the Pakistan meteorological department for the city of Multan.

Auto Regressive Integrated Moving Average (ARIMA)

The acronym ARIMA stands for auto regressive integrated moving average. It is basically generalization of ARMA class of Models and used in the situation where the observed time series is non-stationary. To make the observed series stationary we apply ARIMA class of models where term “I” called integrated term and represents the order of difference that is used to make the series stationary. Mathematically form of ARIMA = ARIMA (p, d, q) Where, P = is the order of a.r terms d = is the order of differencing to make the series stationary q = is the order of m.a terms.

Use of SARIMA Model

Because Maximum temperature is a seasonal phenomenon with twelve months period that’s why The (SARIMA) will be accustomed for this study. SARIMA is created by increasing seasonal variables to ARIMA models. Mathematical form of SARIMA (p, d, q) (P, D, Q)m. (p, d, q) and (P, D, Q)m, are the non-seasonal and seasonal parts the model respectively represent the number of season. Because the data utilized in this study is monthly data with a period of 12 month, we will set the value of m to 12.

Box Jenkins Methodology

Box Jenkins methodology is model building procedure mostly used in ARMA class of models. Prime objective of this approach is “To built and select the best forecast model among list of candidate’s models”. Box-Jenkins Methodology is a 4-step procedure. (1) Identification (2) Estimation (3) Diagnostic Checking (4) Forecasting Box-Jenkins Methodology is applicable only for stationary time series. Therefore very first step to test the Stationarity of the observed time series.

RESULTS AND DISCUSSION

Detection of Stationarity

To determine whether the underlying historic data of the maximum surface air temperature is stationary or not graphical and statistical testing approach is used as discussed below.

Graphical Analysis

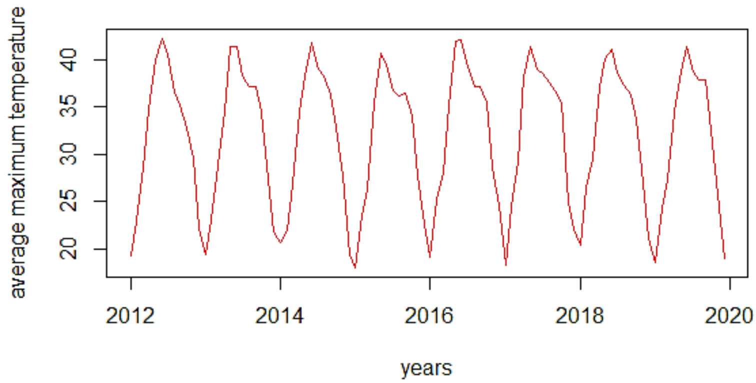


Figure 1: **Time Series Plot of Observed Mean Monthly Maximum Temperature Series**

Monthly average Maximum surface air temperature from January-2012 to December-2019 is plotted in Figure 1. The graphical analysis of monthly average Maximum surface air temperature represents that there is no trend in the given data. However a seasonal component can be observed.

Testing Approach to Assess Stationarity

Table 1
ADF and KPSS tests results of the Observed Monthly Average Maximum Surface Air Temperature

Test Type	Test Statistic	P-value
ADF	-7.9395	0.01
KPSS	0.017545	0.1

Because both tests and graphical analysis shows that the underlying time series data is stationary. Hence it is concluded that observed monthly average maximum surface air temperature is stationary.

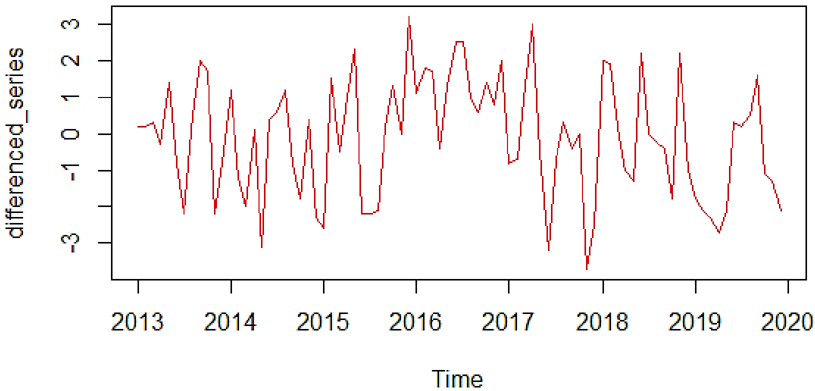


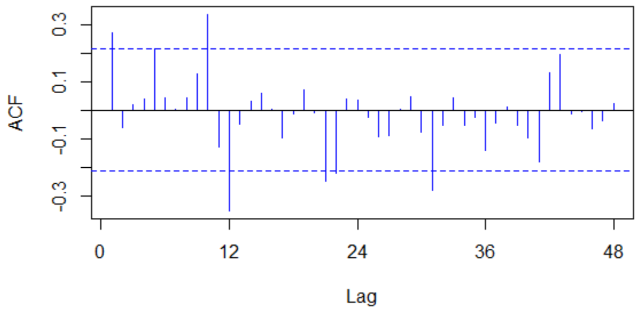
Figure 2: Time Series Plot after First Seasonal Difference of Monthly Average Maximum Surface Air Temperature Series

Seasonal Difference

After ensure the Stationarity and taking the seasonal difference of the original time series now we will apply the formal four steps of Box-Jenkins Methodology.

Identification

In the step of Identification first of all a model is specified among ARMA class of models and then tentative orders of AR and MA terms are identified on the basis of PACF and ACF respectively and finally on the basis of AIC and BIC best model is selected.



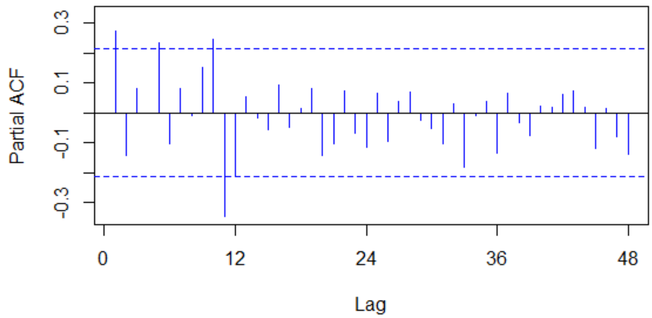


Figure 3: ACF and PACF plot after First Seasonal difference of Monthly Average Maximum Surface Air Temperature Series

From ACF and PACF it is concluded that that SARIMA $(3,0,3)(1,1,1)_{12}$ can be used as a base model.

Selection of Best Model

On the basis of base model there are different SARIMA models are considered with different order as shown in table given blow.

Table 2
Selection of Best Model

Model	AIC	BIC
$(3,0,3)(1,1,1)_{12}$	302.23	324.11
$(1,0,0)(1,1,1)_{12}$	**300.7**	**310.42**
$(1,0,1)(1,1,1)_{12}$	**302.56**	**314.71**
$(2,0,2)(1,1,3)_{12}$	305.83	325.27
$(2,0,0)(0,1,1)_{12}$	302.52	314.67

Above table SARIMA(**(1,0,0)(1,1,1)** and SARIMA **(1,0,1)(1,1,1)$_{12}$** have the lowest AIC and BIC values among list of other candidates models. Hence these two models are considered as competing models. In the final step the forecasting performance of these two competing models will be compared on the basis of some forecasting accuracy measures.

Estimation of SARIMA (1,0,0)(1,1,1)$_{12}$

The parameters of SARIMA(1,0,0)(1,1,1)$_{12}$ are estimated using R-Software and the estimated parameters of SARIMA(1,0,0)(1,1,1)$_{12}$ are given in the following table.

Table 3
Parameters Estimation of SARIMA (1, 0, 0) (1, 1, 1)$_{1}$ Model

Parameters	AR1	SAR1	SMA1
Estimate	0.2587	0.2291	-0.9984
S.E	0.1077	0.1354	1.2033

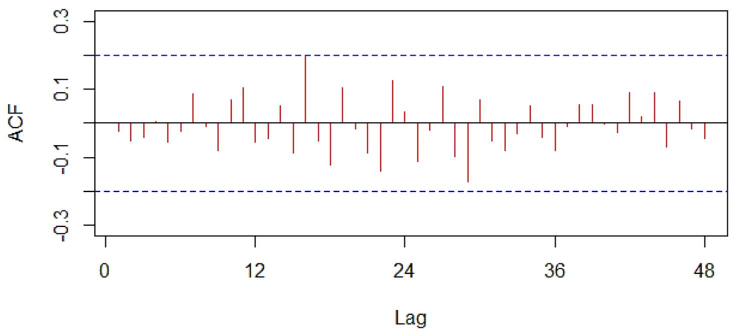
Diagnostic Checking of SARIMA (1,0,0)(1,1,1)$_{12}$
Normality Test for Residuals

Table 4
Shapiro-Wilk and Jarque-Bera tests Results of the Residuals

Test Type	Test Statistic	P-value
Shapiro-Wilk	0.97822	0.1
Jarque-Bera	2.5008	0.3

Because both the tests proved the normality. Hence it can be concluded that residuals of the model **(SARIMA (1, 0, 1)(1, 1, 1)$_{12}$)** follow the assumption of Normality.

Independence/White Noise Testing
ACF Plot of Residuals



In the ACF plot it is observed that all the spikes within confidence band which indicate that residuals are uncorrelated.

Evaluation of Forecasting Accuracy

In this section the accuracy of model SARIMA $(1,0,0)(1,1,1)_{12}$ and SARIMA $(1,0,1)(1,1,1)_{12}$ is compared with the help of graph and on the basis of different accuracy measures such as RMSE, MAE and MAPE.

(a)Graphical Performance of SARIMA $(1,0,0)(1,1,1)_{12}$

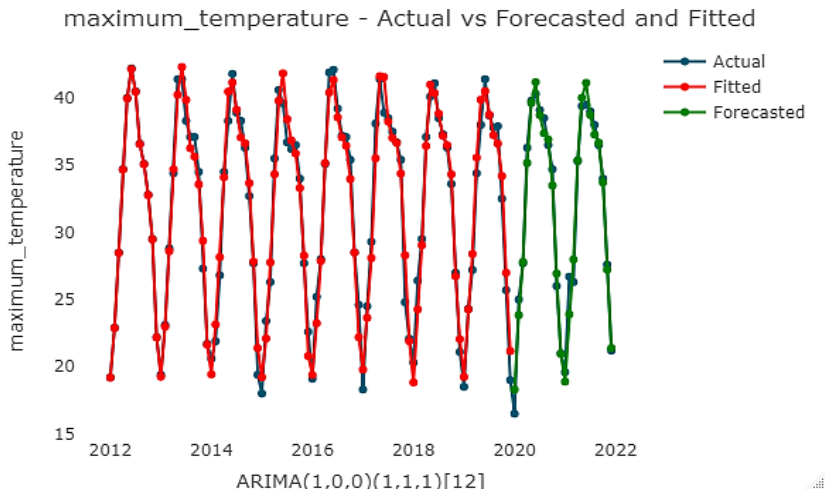


Table 5
In Sample and Out of Sample Forecast Performance

Model	In Sample Forecast Performance			Out of Sample Forecast Performance		
	RMSE	MAE	MAPE	RMSE	MAE	MAPE
$(1,0,0)(1,1,1)_{12}$	1.1464	0.8612	2.9445	1.0174	0.8421	2.7709
$(1,0,1)(1,1,1)_{12}$	1.1475	0.8663	2.9613	1.0300	0.8531	2.8147

Because all the forecasting accuracy measures such as RMSE, MAE and MAPE of model SARIMA**$(1,0,0)(1,1,1)_{12}$** are less than model SARIMA $(1,0,1)(1,1,1)_{12}$ for both cases in sample and for out of sample. Hence it is concluded that on the basis of our study the best forecasting model for Monthly Average Maximum Surface Air Temperature for Multan is SARIMA**$(1,0,0)(1,1,1)_{12}$**.

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AN EXTENSION OF BETA FUNCTION ON THE BASIS OF AN EXTENDED MITTAG-LEFFLER FUNCTION AND ITS APPLICATIONS

Sumbal Rasheed, Ghulam Farid, Hina Saleem, Madiha Ghamkhar
Khurrem Shehzad, Ali Hassan and **Maliha Batool**
Department of Mathematics and Statistics, University of Agriculture
Faisalabad, Pakistan
Email: sunduskanwal6188@gmail.com
ghulamfarid@uaf.edu.pk
hinasaleemuaf@gmail.com
madiha-gm@yahoo.com
khurremshehzad@gmail.com
iamalibajwa@gmail.com
malihabatool1104@gmail.com

ABSTRACT

The purpose of present paper is to introduce a new extension of Beta function that involve the product of four-parametric Mittag-Leffler function in its kernal. Moreover, we discussed about a number of properties like Integral representation, Mellin transformation and Summation formulas related to extended function.

KEYWORDS

Mittag-Leffler function (M-L function), Gamma function, Beta function, pochhammer k-symbol and k-gamma function.

I. INTRODUCTION

The most well-known Beta function, in classical form is given by [17]

$$B(\phi,\psi)=\int_0^1 z^{\phi-1}(1-z)^{\psi-1}dz\,,\ \ \mathrm{Re}(\phi)>0,\mathrm{Re}(\psi)>0 \tag{1}$$

$$B(\phi,\psi)=\frac{\Gamma(\phi)\Gamma(\psi)}{\Gamma(\phi+\psi)},\ \ \mathrm{Re}(\phi)>0,\mathrm{Re}(\psi)>0 \tag{2}$$

the notation $\Gamma(\phi)$ represents the classical Gamma function which is given in [17]

$$\Gamma(\phi)=\int_0^\infty t^{\phi-1}e^{-t}dt,\quad (\mathrm{Re}(\phi)>0) \tag{3}$$

also k-gamma and k-Beta functions given by [18] defined as

$$\Gamma_k(\gamma)=\int_0^\infty w^{\gamma-1}e^{-\frac{\gamma^k}{k}}dw \tag{4}$$

and

$$B_k(\phi,\psi)\int_0^\infty u^{\phi-1}(1-u^k)^{-\frac{\phi+\psi}{k}}\,du \tag{5}$$

In 1903, the function was first presented by Gosta Mittag-Leffler [12] denoted by $E_\alpha(t)$ as

$$E_\alpha(t)=\sum_{n=0}^{\infty}\frac{t^n}{\Gamma(\alpha n+1)},\ \alpha>0\ ,\ t\in \mathrm{C}. \tag{6}$$

$$E_1(t)=e^t\ .$$

In 1905, Wiman introduced a generalization of Mittag-Leffler function $E_{\alpha,\beta}(t)$, that is also known as Wiman function, and is given by [19]

$$E_{\alpha,\beta}(t)=\sum_{n=0}^{\infty}\frac{t^n}{\Gamma(\alpha n+\beta)},\ (\alpha,\beta\in \mathrm{C},\ \mathrm{Re}(\alpha)>0,\ \mathrm{Re}(\beta)>0) \tag{7}$$

$$E_{\alpha,1}(t)=E_\alpha(t)$$

Prabhakar [14] has also introduced a generalization of Mittag-Leffler function, as

$$E^{\gamma}_{\alpha,\beta}(t)=\sum_{n=0}^{\infty}\frac{(\gamma)_n}{\Gamma(\alpha n+\beta)}\frac{t^n}{n!},(\alpha,\beta,\gamma\in \mathrm{C},\mathrm{Re}(\alpha)>0) \tag{8}$$

$$E^{1}_{\alpha,\beta}(t)=E_{\alpha,\beta}(t)$$

Not long ago, a number of extensions related to Beta function have introduced (see [1][3][5][9][10][11][13][16]). Here we discussed some of them:

The following extension is given by Chaudhry et al. [7]

$$B(\alpha,\beta;t)=\int_0^1 u^{\alpha-1}(1-u)^{\beta-1}\exp(\frac{-t}{u(1-u)})du,\mathrm{Re}(t)>0,\mathrm{Re}(\alpha)>0,\mathrm{Re}(\beta)>0 \tag{9}$$

and extended Beta function presented by Choi et al. [8]

$$B(\alpha,\beta;t;w)=\int_0^1 u^{\alpha-1}(1-u)^{\beta-1}\exp\left(\frac{-t}{u}-\frac{w}{1-u}\right)du \tag{10}$$
$$\mathrm{Re}(t)>0,\mathrm{Re}(w)>0,\mathrm{Re}(\alpha)>0,\mathrm{Re}(\beta)>0$$

Subsequently, the following extended function is given by Rahman et al. [15]:

$$B^{\gamma}_{t,w}(\alpha,\beta)=\int_0^1 u^{\alpha-1}(1-u)^{\beta-1}E_\gamma\left(\frac{-t}{u}\right)E_\gamma\left(\frac{-w}{1-u}\right)du \tag{11}$$
$$\mathrm{Re}(\alpha)>0,\mathrm{Re}(\beta)>0,\gamma>0,t,w>0$$

After this, Al-Gonah and Mohammed [2] presented the following extension of extended Beta function:

$$B_t^{(\phi,\psi,\gamma)}(\alpha,\beta)=\int_0^1 u^{\alpha-1}(1-u)^{\beta-1}E_{\phi,\psi}^{\gamma}\left(\frac{-t}{u(1-u)}\right)du, \mathrm{Re}(\alpha)>0, \mathrm{Re}(\beta)>0 \tag{12}$$

Atash et al. [4] have given the following extension of Beta function:

$$B_{t,w}^{(\phi,\psi)}(\alpha,\beta)=\int_0^1 u^{\alpha-1}(1-u)^{\beta-1}E_{\phi,\psi}\left(\frac{-t}{u}\right)E_{\phi,\psi}\left(-\frac{w}{(1-u)}\right)du, \tag{13}$$
$$(\mathrm{Re}(\alpha)>0, \mathrm{Re}(\beta)>0, \mathrm{Re}(\phi)>0, t,w\geq 0)$$

Later, an extension of Beta function is given by Barahmah [6]:

$$B_{t,w}^{(\gamma;\phi,\psi)}(\alpha,\beta)=\int_0^1 u^{\alpha-1}(1-u)^{\beta-1}E_{\phi,\psi}^{\gamma}\left(\frac{-t}{u}\right)E_{\phi,\psi}^{\gamma}\left(-\frac{w}{(1-u)}\right)du, \tag{14}$$
$$(\mathrm{Re}(\alpha)>0, \mathrm{Re}(\beta)>0, \mathrm{Re}(\phi)>0.t,w\geq 0)$$

Here in the present paper, we introduced a new generalization of Beta function with four parametric M-L function that satisfies the classical definition of Beta function in particular cases:

$$B_{t,w}^{(\gamma,k;\phi,\psi)}(\alpha,\beta)=\int_0^1 u^{\alpha-1}(1-u)^{\beta-1}E_{\phi,\psi}^{\gamma,k}\left(\frac{-t}{u}\right)E_{\phi,\psi}^{\gamma,k}\left(-\frac{w}{(1-u)}\right)du \tag{15}$$

where

$$E_{\phi,\psi}^{\gamma,k}(t)=E(\phi,\psi;\gamma,k;t)=\sum_{n=0}^{\infty}\frac{(\gamma)_{n,k}t^n}{\Gamma_k(\phi n+\psi)}, \phi,\psi,\gamma\in \mathrm{C}, \mathrm{Re}(\alpha)>0 \tag{16}$$
$$(\gamma)_{n,k}=\gamma(\gamma+k)(\gamma+2k)...(\gamma+(n-1)k)$$

and

$$\Gamma_k(\gamma)=\int_0^{\infty} w^{\gamma-1}e^{-\frac{\gamma^k}{k}}dw \tag{17}$$

For (16) and (17), See [19].

Declaration 1.

a) Equation (15) satisfy equation (14), For $k=1$

b) Equation (15) satisfy equation (13), For $k=\gamma=1$

c) For $k=\varphi=\psi=\gamma=1$ and t=w, Equation (15) reduces to classical Beta function (1).

II. PROPERTIES

Here in this scenario, we will discuss about some certain results related to new extended Beta function along with Integral form, Summation formulas and Mellin Transform.

Integral Form:

Theorem I

The following integral form holds true:

(i) $$B_{t,w}^{(\gamma,k;\phi,\psi)}(\alpha,\beta)=2\int_0^{\frac{\pi}{2}}\cos^{2\alpha-1}\theta\sin^{2\beta-1}\theta E_{\phi,\psi}^{\gamma,k}\left(-\frac{t}{\cos^2\theta}\right)E_{\phi,\psi}^{\gamma,k}\left(-\frac{w}{\sin^2\theta}\right)d\theta \tag{18}$$

Proof:

Substituting $u=\cos^2\theta$ in (15), we obtain the desire result.

(ii) $$B_{t,w}^{(\gamma,k;\phi,\psi)}(\alpha,\beta)=\int_0^{\infty}\frac{z^{\alpha-1}}{(1+z)^{\alpha+\beta}}E_{\phi,\psi}^{\gamma,k}\left(-\frac{t(1+z)}{z}\right)E_{\phi,\psi}^{\gamma,k}\left(-w(1+z)\right)dz \tag{19}$$

Proof:

For proving (19), we substitute $u=\frac{z}{1+z}$ in (15).

(iii) $$\begin{aligned}B_{t,w}^{(\gamma,k;\phi,\psi)}(\alpha,\beta)=(c-f)^{1-\alpha-\beta}\int_f^c(z-f)^{\alpha-1}(c-z)^{\beta-1}\\ E_{\phi,\psi}^{\gamma,k}\left(-\frac{t(c-f)}{(z-f)}\right)E_{\phi,\psi}^{\gamma,k}\left(-\frac{w(c-f)}{(c-z)}\right)dz\end{aligned} \tag{20}$$

Proof:

After putting $u=\frac{z-f}{c-f}$ in (15), proof of (20) can be obtained.

Theorem II

The following integral formula holds true:

$$\begin{gathered}\Gamma_0^{\phi,\psi,\gamma,k}(u)\Gamma_0^{\phi,\psi,\gamma,k}(v)=\frac{1}{B(x+u,y+v)}\int_0^{\infty}\int_0^{\infty}p^{u-1}q^{v-1}B_{p,q}^{(k,\gamma;\phi,\psi)}(x,y)dpdq,\\ (\mathrm{Re}(x+u)>0,\mathrm{Re}(y+v)>0,\mathrm{Re}(u),\mathrm{Re}(v)>0)\end{gathered} \tag{21}$$

Proof:

Since by [16], we have

$$\begin{gathered}\Gamma_p^{(\phi,\psi,\gamma)}(u)=\int_0^{\infty}t^{u-1}E_{\phi,\psi}^{\gamma}(-t-\frac{p}{t})dt,\\ (\mathrm{Re}(u)>0,\mathrm{Re}(\phi)>0,\mathrm{Re}(p)>0)\end{gathered} \tag{22}$$

So, we obtain

$$\Gamma_0^{(\phi,\psi,\gamma,k)}(u)\Gamma_0^{(\phi,\psi,\gamma,k)}(v)=\int_0^{\infty}w^{u-1}E_{\phi,\psi}^{\gamma,k}(-w)dw\int_0^{\infty}z^{v-1}E_{\phi,\psi}^{\gamma,k}(-z)dz$$

$$=\frac{B(x+u,y+v)}{B(x+u,y+v)}\int_0^{\infty}\int_0^{\infty} w^{u-1}z^{v-1}E_{\phi,\psi}^{\gamma,k}(-w)E_{\phi,\psi}^{\gamma,k}(-z)dwdz$$

$$=\frac{1}{B(x+u,y+v)}\int_0^1 t^{(x+u)-1}(1-t)^{(y+v)-1}\left\{\int_0^{\infty}\int_0^{\infty} w^{u-1}z^{v-1}E_{\phi,\psi}^{\gamma,k}(-w)E_{\phi,\psi}^{\gamma,k}(-z)dwdz\right\}dt$$

putting $w=\frac{p}{t}$ and $w=\frac{q}{1-t}$, we obtain

$$\Gamma_0^{(\phi,\psi,\gamma,k)}(u)\Gamma_0^{(\phi,\psi,\gamma,k)}(v)=\frac{1}{B(x+u,y+v)}\int_0^{\infty}\int_0^{\infty} w^{u-1}z^{v-1}$$

$$\left\{\int_0^1 t^{x-1}(1-t)^{y-1}E_{\phi,\psi}^{\gamma,k}\left(\frac{-p}{t}\right)E_{\phi,\psi}^{\gamma,k}\left(-\frac{q}{1-t}\right)dt\right\}dwdz$$

$$\Gamma_0^{\phi,\psi,\gamma,k}(u)\Gamma_0^{\phi,\psi,\gamma,k}(v)=\frac{1}{B(x+u,y+v)}\int_0^{\infty}\int_0^{\infty} w^{u-1}z^{v-1}B_{w,z}^{(k,\gamma;\phi,\psi)}(x,y)dwdz$$

- The desire result match with well-known result of Atash et al. [13] if we put $\gamma=k=1$
- For $k=1$, equation (21) reduces to a so-called result of Barahmah [17].

Summation Formulas:

Theorem III

The summation forms given below satisfy:

$$B_{t,w}^{(\gamma,k;\phi,\psi)}(\alpha,1-\beta)=\sum_{j=0}^{\infty}\frac{(\beta)_j}{j!}B_{t,w}^{(\gamma,k;\phi,\psi)}(\alpha+j,1) \tag{23}$$

Proof:

From (15), we obtain

$$B_{t,w}^{(\gamma,k;\phi,\psi)}(\alpha,1-\beta)=\int_0^1 u^{\alpha-1}(1-u)^{-\beta}E_{\phi,\psi}^{\gamma,k}\left(\frac{-t}{u}\right)E_{\phi,\psi}^{\gamma,k}\left(-\frac{w}{(1-u)}\right)du$$

By using generalized Binomial theorem

$$(1-u)^{-\beta}=\sum_{j=o}^{\infty}\frac{(\beta)_j}{j!}u^n,|u|<1$$

we have the desired result.

Theorem IV

Given below form holds true:

$$B_{t,w}^{(\gamma,k;\phi,\psi)}(\alpha,\beta)=\sum_{j=0}^{\infty}B_{t,w}^{(\gamma,k;\phi,\psi)}(\alpha+j,\beta+1) \tag{24}$$

Proof:

From (15), we have

$$B_{t,w}^{(\gamma,k;\phi,\psi)}(\alpha,\beta)=\int_0^1 u^{\alpha-1}(1-u)^{\beta-1}E_{\phi,\psi}^{\gamma,k}\left(\frac{-t}{u}\right)E_{\phi,\psi}^{\gamma,k}\left(-\frac{w}{(1-u)}\right)du$$

using generalized Binomial theorem,

$$(1-u)^{\beta-1}=(1-u)^{\beta}\sum_{j=o}^{\infty}u^{j},|u|<1 \tag{25}$$

after putting R.H.S of (24) and then simplification we have the desire result.

Theorem V

$$\sum_{j=0}^{n}\overset{n}{\underset{j}{C}}\,B_{t,w}^{(\gamma,k;\phi,\psi)}\;(\alpha+j,\beta+n-j)=B_{t,w}^{(\gamma,k;\phi,\psi)}(\alpha,\beta),n\in N_0 \tag{26}$$

Proof:

We use Mathematical induction method on ($n\in N_0$), as

For n=0, (26) holds trivially.

For n=1, we have

$$=\sum_{j=0}^{1}\overset{1}{\underset{j=0}{C}}\,B_{t,w}^{(\gamma,k;\phi,\psi)}\;(\alpha+j,\beta+1-j)$$

$$=B_{t,w}^{(\gamma,k;\phi,\psi)}\;(\alpha,\beta+1)+B_{t,w}^{(\gamma,k;\phi,\psi)}\;(\alpha+1,\beta)$$

$$=\int_0^1\left\{u^{\alpha-1}(1-u)^{\beta}+u^{\alpha}(1-u)^{\beta-1}\right\}\mathrm{E}_{\phi,\psi}^{k,\gamma}\left(\frac{-t}{u}\right)\mathrm{E}_{\phi,\psi}^{k,\gamma}\left(\frac{-w}{1-u}\right)du$$

$$=\int_0^1 u^{\alpha-1}(1-u)^{\beta-1}\mathrm{E}_{\phi,\psi}^{k,\gamma}\left(\frac{-t}{u}\right)\mathrm{E}_{\phi,\psi}^{k,\gamma}\left(\frac{-w}{1-u}\right)du$$

$$=B_{t,w}^{(\gamma,k;\phi,\psi)}(\alpha,\beta)$$

So, it is true for n=1 and continuing the same process $\forall n\in N_0$, we finally have the desired result i.e. (26).

Theorem VI

The following Mellin transform satisfies:

$$B_{t,w}^{(\gamma,k;\phi,\psi)}(\alpha,\beta)=\frac{1}{(2\pi j)^2}\int_{\gamma_1-j\infty}^{\gamma_1+j\infty}\int_{\gamma_2-j\infty}^{\gamma_2+j\infty}B(\alpha+u,\beta+v)\Gamma_0^{\phi,\psi,\gamma,k}(u)\Gamma_0^{\phi,\psi,\gamma,k}(v)t^{-r}w^{-s}dudv,$$

$$(Re(\alpha)>0,\mathrm{Re}(\beta)>0,\mathrm{Re}(\phi)>0,\mathrm{Re}(\psi)>0,t,w\geq 0,\gamma_1>0,\gamma_2>0) \tag{27}$$

Proof:

By applying the definition of Mellin Transform to the R.H.S of (21), we have

$$\mathrm{M}\{B_{t,w}^{(\gamma,k;\phi,\psi)}(\alpha,\beta);t\to u,w\to v\}$$

$$=B(\alpha+u,\beta+v)\Gamma_0^{\phi,\psi,\gamma,k}(u)\Gamma_0^{\phi,\psi,\gamma,k}(v),$$
$$\mathrm{Re}(\alpha+u)>0,\mathrm{Re}(\beta+v)>0,\mathrm{Re}(\phi)>0,\mathrm{Re}(\psi)>0.$$

So, by taking inverse Mellin transform on both sides, desire result can achieve.

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FERTILITY MODELING THROUGH HADWIGER FUNCTION: AN APPLICATION TO AGE SPECIFIC FERTILITY RATES IN PAKISTAN

Hira Fatima Waseem[1,2§] and **Farah Yasmeen**[1]
[1] Department of Statistics, University of Karachi, Karachi, Pakistan
[2] School of Public Health, Dow University of Health Sciences
Karachi, Pakistan. Email: riazfarah@yahoo.com
[§] Corresponding author Email: hira.waseem@duhs.edu.pk

ABSTRACT

The modeling of fertility curves has attracted the demographers for many years. Fertility modeling is main concerned for studying the reproductivity and the modeling of fertility curves by theoretical distribution has been used to describe reproductive patterns of women in childbearing age.

Various types of models have been proposed in literature to capture the fertility patterns. This paper presents the application of Hadwiger Function (HWF) model on age specific fertility rates (ASFR) in Pakistan. The application of HWF model provides meaningful demographic interpretation of its parameters.

In this paper, we try to fit the basic HWF model on ASFR of Pakistan during the period 1984-2007. This paper briefly explains the application of HWF and to estimate and interpret the model parameters.

KEYWORDS

Hadwiger function, fertility pattern, fertility modeling, age specific fertility rates.

1. INTRODUCTION

Fertility refers to reproductive performance of woman or a group of women and is considered to be one of the main population characteristics (Nasir et al. 2010). It is of chief interest to statistician and demographers.

Pakistan is the world's sixth most populous country with an estimated 207.77 million inhabitants (Pakistan Bureau of Statistics, 2017). During 1960s to mid-1980s, fertility rates in Pakistan remained more or less stable with more than 6 children per woman and then started to decline in the late 1980s (Sathar and Casterline 1998). During the period 1980-2006, it was estimated that the fertility in Pakistan declined around 40% (World Bank, 2010). Estimates from different sources imply decline in fertility in Pakistan particularly after 1990s due to introduction of family planning scheme, rising age at marriage and contraceptive awareness programs in that period (Hakim and Mahmood 1994, Sathar and Casterline 1998).

The age-specific fertility rates (ASFR) pattern of Pakistan shows a typical shape. In order to describe this typical shape of ASFR, both parametric and non-parametric models have been reported for years. They include Keyfitz (1967) and Nurul-Islam and Mallick (1987).

Hadwiger (1940) and Gilje (1969) described that a simple Hadwiger function can be well fitted on the distributions of ASFR in most Continental European populations. Chandola *et al*. (1999) has reported that Hadwiger function is more suited for fertility curves. Islam and Ali (2004) and Nasir *et al.* (2009) reported that the third degree polynomial worked as best model to fit the ASFR. Both studies used polynomial model but they did not provide the interpretation of model parameters.

Zakria *et al.* (2013) applied two interpretable parsimonious models; Hadwiger function model and Peristera and Kostaki (P-K) model-I on ASFR of Pakistan for the year 2007 only. Yasmeen and Mahmood (2012) applied functional time series (FTS) model of Hyndman and Ullah (2007) to fertility rates of Pakistan. Vähi (2017) studied the fertility curves of Estonian fertility data by using the Hadwiger distribution. However, there is a need to develop the model to fit on fertility rates which provide a better fit with meaningful interpretation of its parameters.

In this paper, we apply Hadwiger function model to ASFR of Pakistan. This mathematical model provides description of some fertility indices through its interpretable parameters. This paper is distributed as follows. In above section 1, the introduction and literature regarding fertility modeling and fertility pattern in Pakistan are discussed. In section 2, the source of fertility data and methodology of HWF model are discussed and the results are obtained and described in section 3. Finally, some concluding remarks are given in section 4.

2. MATERIAL AND METHODS

2.1 Data from Pakistan Demographic Surveys (PDS)

The secondary data of annual age specific fertility rates of Pakistan for ages 15-49 were obtained from Pakistan demographic survey (PDS). These data are available for the years 1984-1986, 1988-1992, 1995-1997, 1999-2001, 2003 and 2005-2007. Age-specific fertility rates are missing for rest of the years, so we estimate those using interpolating splines. For each age-group, we consider a fixed number of knots as available ASFR and estimate the function for missing years. The reader is referred to Yasmeen and Mahmood (2012) for available and estimated age-specific fertility rates (ASFR).

2.2 Hadwiger Function (HWF) Model

The main Hadwiger function (HWF) model (Hadwiger 1940; Gilje, 1969) is given by

$$f\left(x\right)=\frac{ab}{c}\left(\frac{c}{x}\right)^{\frac{3}{2}}\exp\left\{-b^{2}\left(\frac{c}{x}+\frac{x}{c}-2\right)\right\} \tag{1}$$

where

x represents age of the mother at the time of birth

a, b and c are the parameters to be estimated from the observed fertility data

Demographic interpretation of these parameters is given below;

- a is associated with total fertility,
- b determines the height of the curve
- c is related to the mean age of motherhood,
- $ab|c$ is associated to the maximum age-specific fertility rate.

2.3 Fitting Hadwiger Function (HWF) Model to Fertility Rates in Pakistan

In this section, we apply Hadwiger function model on ASFR of Pakistan. The parameters of the HWF model are determined for the fertility rates in observed years (1984-2007). By using the estimates of parameters, the fitted Hadwiger curves are obtained and then compared with the observed (original) fertility curves. The statistical analysis for fitting the HWF model is performed in R version 3.5.2 with the package ‘fmsb’ available at CRAN.

3. RESULTS

Figure 1 displays the age specific fertility curves for selected years (1985, 1990, 1995, 2000 and 2005). The order of colors (from earliest to latest) is red, yellow, green, violet and blue, which showed a typical inverse V-shape pattern of fertility curves. This figure shows that fertility rates have been decreased for all age-groups but greater decline was observed in the middle age group i.e. 20 to 34 years while slower decline in the other age groups during the past 23 years (1984-2007). The reason for this fertility decline might be that the government introduced family planning schemes and contraceptive awareness programs in the year 1990s.

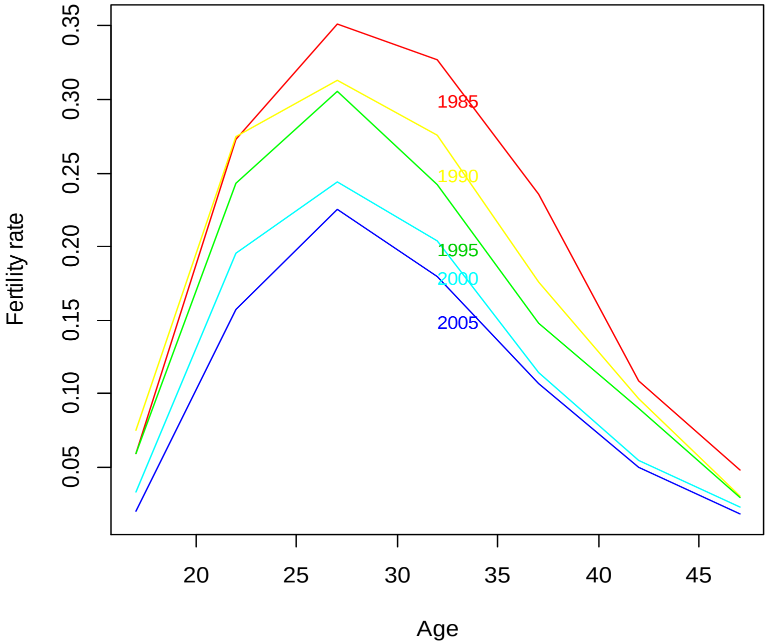


Figure 1: Age-Specific Fertility Rates of Pakistan for Selected Years

Table 1 presents the estimates of parameters and root mean square error (RMSE) values of HWF model applied on fertility rates of Pakistan (1984-2007).

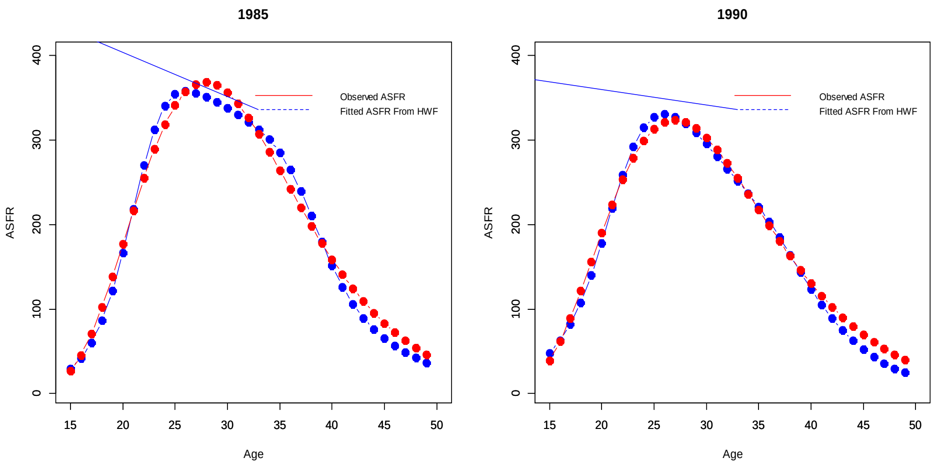
Table 1
Parameter Estimates of Hadwiger Function (HWF) Model for ASFR of Pakistan during 1984-2007

Years	Parameter Estimates			
	a	b	c	ab/c
1984	4124.57	2.56	31.04	339.90
1985	4167.00	2.54	31.32	337.86
1986	4125.57	2.51	31.39	329.80
1987	4098.65	2.57	30.95	339.84
1988	3887.01	2.43	31.12	303.66
1989	3862.63	2.37	31.05	295.19
1990	3737.79	2.41	30.65	293.85
1991	3576.45	2.49	30.48	291.89
1992	3455.50	2.52	30.02	289.96
1993	3321.74	2.51	30.34	274.80
1994	3293.53	2.52	29.93	277.30
1995	3255.50	2.52	29.55	277.63
1996	3198.72	2.69	29.69	289.26
1997	2969.54	2.55	29.97	252.85
1998	2766.07	2.59	29.99	239.29
1999	2623.81	2.75	29.72	242.78
2000	2506.57	2.80	29.75	236.20
2001	2373.66	2.90	30.19	227.92
2002	2302.52	2.93	30.22	223.34
2003	2269.18	2.91	29.97	220.02
2004	2222.89	2.91	29.84	217.10
2005	2177.71	2.94	29.89	214.09
2006	2147.04	2.96	30.02	211.63
2007	2109.68	2.99	29.88	211.16

The results of estimated parameters of HWF model can be easily interpreted. For the year 1984, the estimated parameters of HWF model shows that the total fertility is 4124.57 per thousand women, the model age of mothers is 31.04 years and estimated maximum ASFR is 339.90 per thousand women (the value of ab/c). For the year 1985, the estimated parameter ‘a’ is 4167 which is associated with total fertility per thousand women. The parameter ‘c’ is 31.32 years as the estimated modal age of motherhood and 337.86 as the estimated maximum ASFR per thousand women obtained by (ab/c). The same demographic interpretation can be followed for rest of the years (1984-2007) as shown in Table 1.

3.1 Estimated Fertility Curves from HWF Model

By using the parameter estimates of HWF model, the fertility curves are modeled. The observed fertility curve and HWF model curves are plotted in Figure 2 for selected years 1985, 1990, 1995, 2000 and 2005. These figures showed that the observed and fitted ASFR are very similar in shape and values at younger and older ages while slightly vary in middle age groups. It shows that the HWF model is as good approximation to the ASFR of Pakistan, especially for the recent years.



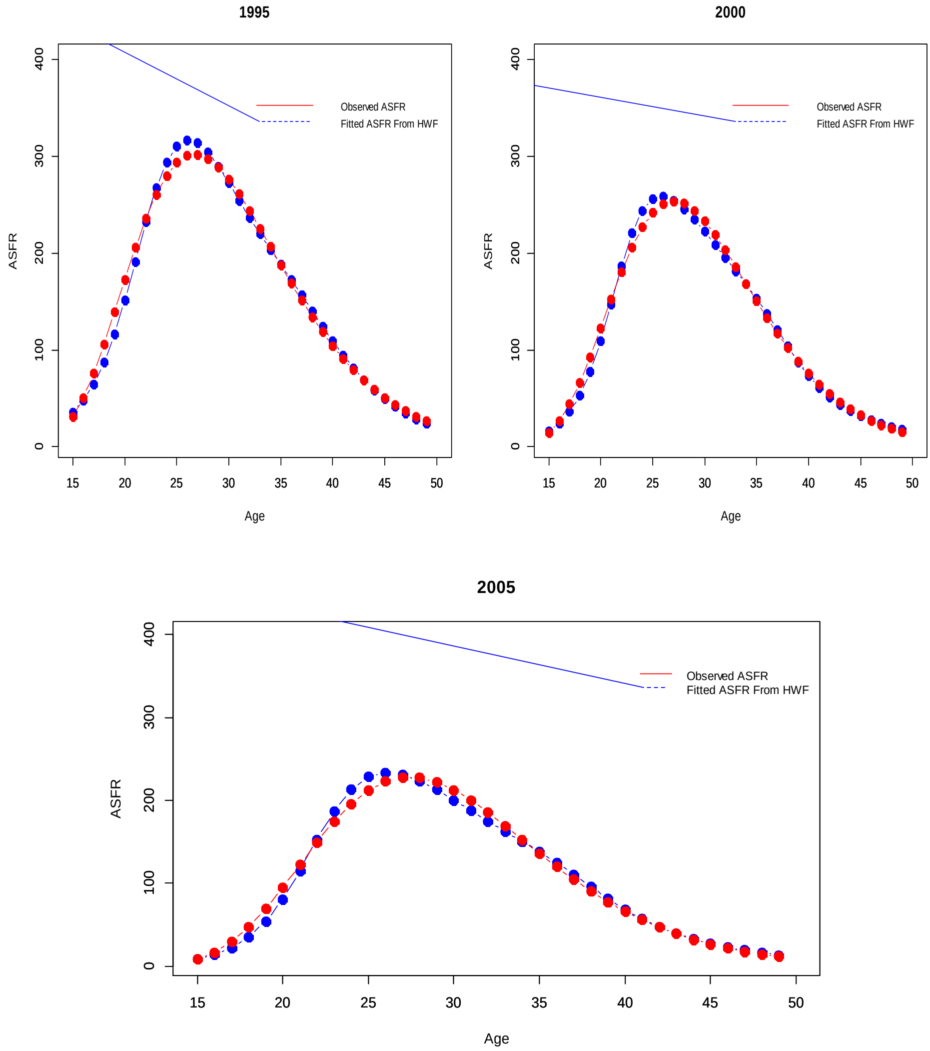


Figure 2: Observed and Estimated ASFR in Pakistan from HWF during 1985, 1990, 1995, 2000 and 2005

4. COMMENTS AND CONCLUSION

The overall conclusion is that Hadwiger function provides the better estimation and demographic interpretation of parameters (TFR, model age of mother hood and maximum ASFR values) for fertility in Pakistan. The findings of this paper suggest that the simple Hadwiger model may be useful for describing fertility trends of different countries. This paper concludes that some demographic features are captured by the HWF model on fertility data of Pakistan.

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INTEGRAL TRANSFORMS OF SOME GENERALIZED FUNCTIONS

Hafiza Maliha Batool, Ghulam Farid, Ali Hassan and **Mohib Ali**
Department of Mathematics and Statistics,
University of Agriculture Faisalabad, Pakistan
Email: malihabatool1104@gmail.com
ghulamfarid@uaf.edu.pk
iamalibajwa@gmail.com
amohib319@gmail.com

ABSTRACT

Integral transform is the most important method that are used to solve the problem in applied mathematics, Engineering science and Mathematical physics which are determined by integral equations, difference equations and differential equations. The purpose of present paper is to introduce some generalized function. In this paper, some integral transform to hypergeometric functions will be solved. Moreover, we discussed about the results in the form of generalized Wright function which are appropriate in different extent of sciences.

KEYWORDS

Generalized function, Special R-function, Integral transforms, Generalized Wright Hypergeometric function.

Subject Classification: 35A22, 26A33, 5632, 41A30.

1. INTRODUCTION

Integral transform is a mathematical operator that creates a new function by multiplying two existing functions. Integral transformation is the name of this process which is represented by

$$F(y)=\int_{-\infty}^{\infty} K(x,y)F(x)dx.$$

An Integral transform is useful if it allows one to turn a complicated problem into a simpler one. A number of transformations are named after the mathematicians who first proposed them (William, 2016).

Generalized function are continuous linear functional that exist in a space of infinitely differentiable functions and have dervatives that are generalized function as well. The delta function is by for the most frequent generalized function for (Kabra and Nagar, 2019).

Lorenzo and Hartley have defined the special function named as

$$G_{\rho,\eta,t}[p,q]=z^{t\rho-\eta-1}\sum_{m=o}^{\infty}\frac{(t)_n\left(pq^{\rho}\right)^m}{\Gamma(n\rho+\rho t-\eta)m!},\ \mathrm{Re}(\rho t-\eta) \tag{1}$$

The above special function is defined by Lorenzo and Hartly.

In this paper the generalized function is defined as

$$G_{k,\ \rho,\eta,t}^{(k)}[p,q]=z^{k(t\rho-\eta)-1}\sum_{m=o}^{\infty}\frac{(t)_{n,k}\left(pq^{k\rho}\right)^m}{\Gamma_k(n\rho+\rho t-\eta)m!},\ \mathrm{Re}(\rho t-\eta)$$

Euler transformation of the function $f(t)$ is described by Sneddon (1979).

$$E\{f(t);ka,kb\}=\int_0^1 t^{ka-1}(1-t)^{kb-1}f(t)dt,\ a,b\in C,R(a)>0,R(b)>0. \tag{2}$$

Euler transform is a term used to illustrate the connection between two functions (Kabra and Nagar, 2019).

Generalized Hypergeometric Function is defined by Srivastava and Karlson (1985)

$${}_sF_t\left[\begin{matrix}\alpha_{1,\dots,}\ \alpha_s; & \\ & z\\ \beta_{1,\dots,}\ \beta_t; & \end{matrix}\right]=\sum_{m=0}^{\infty}\frac{(\alpha_1)_m\cdots(\alpha_p)_m}{(\beta_1)_m\cdots(\beta_q)_m}\frac{z^m}{m!}\ , \tag{3}$$

It is known as the Generalized Gauss series or Generalized Hypergeometric function Rainville (1960) The numerator and denominator parameters take on complex values if s and t are positive integers or zero, and the numerator and denominator parameters are both positive integers (Kabra and Nagar, 2019).

Some compositions are expressed as generalized Wright Hypergeometric function ${}_m\psi_n(z)$ for detail, see Wright (1935), for $z\in C,s_i,t_j\in C$ and $\alpha_i,\beta_j\in R/\{0\}$,

$${}_m\psi_n(z)={}_m\psi_n(z)\left[\begin{matrix}(s_i,\alpha_i)_{1,m} & \\ & ;yz\\ (t_j,\beta_j)_{1,n} & \end{matrix}\right]=\sum_{k=0}^{\infty}\frac{\prod_{i=}^{m}\Gamma(s_i+\alpha_i)z^k}{\prod_{j=}^{n}\Gamma(t_j+\beta_j)k!} \tag{4}$$

Whittaker transformation is described as Whittaker and Watson, (1996):

$$\int_0^{\infty}e^{\frac{-ks}{2}}s^{\psi-1}W_{k\ \alpha,\beta}^{(k)}(s)\,ds=\frac{\Gamma_k\left(\frac{1}{2}+\beta+\psi\right)\Gamma_k\left(\frac{1}{2}-\beta+\psi\right)}{\Gamma_k(1-\alpha+\psi)}\ ,$$

where $R(\beta\pm\psi)>\frac{-1}{2}$ and $W_{\alpha,\beta}(s)$ is named as the confluent Hypergeometric function.

2. INTEGRAL TRANSFORMS OF GENERALIZED FUNCTION $G^{(k)}_{k,\rho,\eta,s}[a,b]$

Theorem I:

$$E\{G^{(k)}_{k,\ \rho,\eta,r}[p,q];kp,kq\}=\frac{\Gamma_k(q)}{\Gamma_k(r)}\,{}_2\psi_2\left[\begin{matrix}(r,1)_k,(r\rho+p-\eta-1,\rho)_k \\ & ;a \\ (r\rho-\eta,\rho)_k,(r\rho+p-\eta-1,\rho)_k\end{matrix}\right],$$

Proof:

By using Euler transform (3), we get

$$E(G^{(k)}_{k,\ \rho,\eta,r}[a,b];kp,kq)=\int_0^1 t^{kp-1}(1-t)^{kq-1}\,G^{(k)}_{k,\ \rho,\eta,r}[a,b]dt$$

$$=\sum_{m=o}^{\infty}\frac{\Gamma_k(r+mk)\,a^n}{\Gamma_k(r)\Gamma_k(r\rho+m\rho-\eta)m!}\int_0^1 t^{k(r\rho+m\rho+p-\eta)-1-1}(1-t)^{kq-1}dt$$

$$=\sum_{m=o}^{\infty}\frac{\Gamma_k(r+mk)\,a^m}{\Gamma_k(r)\Gamma_k(r\rho+m\rho-\eta)m!}B_k\left((r\rho+m\rho-\eta-1,q)\right)$$

$$=\sum_{m=o}^{\infty}\frac{\Gamma_k(r+mk)\,a^m}{\Gamma_k(r)\Gamma_k(r\rho+m\rho-\eta)m!}\,\frac{\Gamma_k(r\rho+m\rho+p-\eta-1)\Gamma_k(q)}{\Gamma_k(q+r\rho+m\rho+p-\eta-1)}$$

$$=\frac{\Gamma_k(q)}{\Gamma_k(r)}\sum_{m=o}^{\infty}\frac{\Gamma_k(r+mk)\,a^m}{\Gamma_k(r\rho+m\rho-\eta)m!}\,\frac{\Gamma_k(r\rho+m\rho+p-\eta-1)}{\Gamma_k(q+r\rho+m\rho+p-\eta-1)}$$

$$E\{G^{(k)}_{k,\ \rho,\eta,r}[a,b];kp,kq\}=\frac{\Gamma_k(q)}{\Gamma_k(r)}\,{}_2\psi_2\left[\begin{matrix}(r,1)_k,(r\rho+p-\eta-1,\rho)_k \\ & ;a \\ (r\rho-\eta,\rho)_k,(r\rho+p-\eta-1,\rho)_k\end{matrix}\right] \tag{5}$$

As required.

Theorem II:

Let $\rho,\eta,r,\in C$ and $R(s)>0$, be such that

$$L\{G^{(k)}_{k,\ \rho,\eta,r}[a,z]=\frac{s^{\eta-r\rho}}{\Gamma_k(r)}\sum_{n=0}^{\infty}\Gamma_k(r+nk)\left(\frac{a}{s^{k\rho}}\right)^n$$

Proof:

On using (1) and (4), we get

$$L\{G_{k,\ \rho,\eta,r}^{(k)}[a,z];ks\}=\int_0^\infty e^{-ksz}G_{k,\ \rho,\eta,r}^{(k)}[a,z]dz$$

$$=\sum_{n=o}^{\infty}\frac{(r)_{n,k}a^n}{\Gamma_k(r\rho+n\rho-\eta)n!}\int_0^\infty z^{k(r\rho+n\rho+p-\eta)-1}e^{-ksz}dz$$

On using (7)

$$=\sum_{n=o}^{\infty}\frac{(r)_{n,k}a^n}{\Gamma_k(r\rho+n\rho-\eta)n!}\frac{\Gamma_k(r\rho+\eta\rho-\eta)}{s_k^{(r\rho+\eta\rho-\eta)}} \tag{6}$$

Declaration:

The above result satisfied (1) and (2), For k=1

Theorem III:

The following Mellin Tranform holds true for Generalize Function:

$$\int_0^\infty x^{p-1}G_{k,\ \varepsilon,\eta,s}^{(k)}[b,p]dx=\frac{1}{k(p+s\varepsilon+n\varepsilon-\eta-1)}\sum_{n=o}^{\infty}\frac{(s)_{n,k}b^n}{\Gamma_k(n\varepsilon+s\varepsilon-\eta)},$$

$where\ \varepsilon,\eta,s,\in C;R(s)>0.$

Proof:

Using the definition of Mellin Transform which is given as

$$M[f;r]=\int_0^\infty t^{r-1}f(t)dt,\quad R(r)>0 \tag{7}$$

now by using (1) we get

On changing the order, we have

$$=\sum_{n=o}^{\infty}\frac{(s)_{n,k}b^n}{\Gamma_k(s\rho+n\rho-\eta)n!}\int_0^\infty x^{k(s\rho+n\rho-\eta-1)+s-1}dx \tag{8}$$

Put $x=e^{-t}$ we get the required result.

3. CONCLUSION

The results achieved in this study are new, and they may be further adjusted using a variety of new and well-known integral transforms utilized in engineering, applied mathematics, science, and other fields.

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THE INDEFINITE INTEGRAL OF MEIJER'S G-FUNCTION FROM INTEGRATING FACTORS

Abdus Saboor and **Arshid Khan**
Institute of Numerical Sciences, Kohat University of Science & Technology
Kohat, Pakistan

ABSTRACT

Some general integrals are presented which were obtained from two integrating factors $g(x)$ and $g_1(x)$ for the first two and last two terms respectively of the second-order linear ordinary differential equations obeyed by many special functions. The integrating factor g(x) is identical to the Lagrangian factor introduced by Conway in Brychkov (2008) and sometimes provides book integrals. The integrals provided by the second factor seem to have been little explored. Conway (2015) takes confluent hyper geometric function and Gauss hyper geometric function. But we will take Meijer's G-Function which is the further generalization of generalization hyper geometric function. Furthermore we can say that Gauss is will as Kummer confluent hyper geometric function are particular cases of generalization hyper geometric function but generalization hyper geometric function is itself particular case of Meijer's G-Function are given for Meijer's G-Function.

KEYWORDS

Gamma function, Pochhammer symbol, Confluent Hyper geometric function, Gauss hyper geometric function, generalized hyper geometric function and Meijer-G function.

INTRODUCTION

In tables of integrals such as those in [Brychkov (2008), Conway (2015, 2018 and 2020) and Gradshteyn et al. (2007)] the bulk of the content consists of definite integrals, with almost token collections of indefinite integrals. However, the relatively few indefinite integrals which have been published tend to be well known and heavily used. This paper presents a new and surprisingly simple method for deriving indefinite integrals of both elementary and special functions, provided the function satisfies an ordinary linear differential equation. However, the relatively few Indefinite integrals which have been published tend to be well known and heavily used.

Conway (2018) introduced a new and simple method "Lagrangian method" for deriving indefinite integrals of both elementary and special functions, provided the function satisfy second order linear differential equation. Many well-known integrals in the literature can be derived very simply from Lagrangian method. The total number of indefinite integrals for special functions can be increased by a large factor using the method introduced by Conway (2018). In particular, it is possible to derive integrals combining products of different special functions. The Lagrangian method was originally derived from an Euler-Lagrange equation, it is explained as follows.

Let we have a second-order linear differential equation:

1. First Integrating Factor

$$w''(x) + p(x)w'(x) + q(x)w(x) = 0 \tag{1}$$

$$g(x) = e^{\int p(x)dx} \tag{2}$$

Now find the values of p(x) from equation (2) and simplify (1) we get:

$$g(x)q(x)w(x) = [g(x)\, \mathrm{w}'(\mathrm{x})]', \tag{3}$$

by integrating Equation (3), we have:

$$\int \mathrm{g(x)q(x)w(x)dx} = -\, g(x)w'(x). \tag{4}$$

The above integral can be generalized by multiplying both sides of Equation (3) with $m(g(x)w'(\mathrm{x}))^{m-1}$ gives:

$$mq(x)g^{m}(x){w'}^{m-1}(x)w(x) = [(g(x)w'(x))^{m}]', \tag{5}$$

and hence:

$$\int q(x)g^{m}(x){w'}^{m-1}(x)w(x)dx = -\frac{g(x)^{m}{w'}^{m}(x)}{m}. \tag{6}$$

2. Second Integrating Factor

For $p(x) \neq 0$, an alternative integrating factor can be introduced for the last two terms in Equation (1). The Equation (1) can also be expressed in the form:

$$w''(x) + p(x)[w'(x) + \frac{\mathrm{q(x)w(x)]}}{p(x)} = 0, \tag{7}$$

and the integrating factor is:

$$g_1(x) = e^{\int \frac{q(x)}{p(x)}dx} \tag{8}$$

Put the value of Equation (8) in Equation (7) and on simplification, we have:

$$\frac{g_1(x)w''(x)}{p(x)} = [g_1(x)w(x)]', \tag{9}$$

On integration the above equation becomes:

$$\int \frac{\mathrm{g_1(x)w''(x)}}{\mathrm{p(x)}}\mathrm{dx} = -\, g_1(x)w\,(x). \tag{10}$$

This integral equation can be generalized by multiplying Equation (9) with

$m(g_1 w(x))^{\,m-1}\ gives$:

and by integration, we get:

$$\int \frac{\mathrm{g_1(x)^m\ w(x)^{m-1} w''(x)}}{\mathrm{p(x)}}\mathrm{dx} = -\frac{g_1(x)^m w(x)^m}{m}. \tag{11}$$

Conway (2015, 2018 and 2020) used different special functions i.e. Airy function of the first and second kind, Complete elliptic integral of the first and second kind, Modified Bessel function of the first and second kind, Bessel functions of the first and second kind, Legendre function, Associated Legendre function of the first and second kind, Confluent and Gauss hypergeometric function.

We have evaluated some indefinite integrals of Meijer's *G*-function (2018), using the Lagrangian method.

The Meijer's *G*-function is a more generalized form of all special functions including used by Conway (2015, 2018 and 2020).

The definition of Meijer's *G*-function is given below.

$$G_{p,q}^{m,n}\left(a_1, a_2, \dots, a_p, b_1, b_2, \dots, b_q, z\right)$$
$$= \frac{1}{2\pi i} \oint \frac{\prod_{j=1}^{m} \Gamma(bj - s) \prod_{j=1}^{n} \Gamma(1 - aj - s)}{\left(\prod_{j=m+1}^{q} \Gamma(1 - bj + s)\right) \left(\prod_{j=n+1}^{p} \Gamma(aj + s)\right)} z^s \,\mathrm{ds} \tag{12}$$

where each $\mathrm{a_i}(i = 1,2,\dots,p)$ and $\mathrm{b_j}(j = 1,2,\dots,q)$ are complex parameters and $0 \leq m \leq q$ and $0 \leq n \leq p$.

3. Indefinite Integration of Meijer's G-Function when p=1, q=2

We have derived indefinite integration of Meijer's G–function for the case $p = 1$, $q = 2$.

Let us take:

$$W_1(x) = \frac{\Gamma(b)}{\Gamma(a)} G_{1,2}^{1,1}\left(\begin{matrix} 1-a \\ 0,1-b \end{matrix} \middle| -x\right), \tag{13}$$

be a Meijer's G-function which obeys the differential equation:

$$W''_1(x) - \left(\frac{b}{x} - 1\right) W'_1(x) - \frac{a}{x} W_1(x) = 0. \tag{14}$$

For which the integrating factors are:

$$\mathrm{g(x) = x^b e^{-a}}, \tag{15}$$

$$\mathrm{g_1(x) = (b - x)^a}, \tag{16}$$

$$\mathrm{W_1}'(x) = \frac{\Gamma(\mathrm{b})}{\Gamma(\mathrm{a})} \mathrm{G}_{1,2}^{1,1}\left(\begin{matrix} -\mathrm{a} \\ 0, -\mathrm{b} \end{matrix} \middle| -\mathrm{x}\right), \tag{17}$$

$$\mathrm{W_1}'(x) = \frac{\Gamma(\mathrm{b})}{\Gamma(\mathrm{a})} \mathrm{G}_{1,2}^{1,1}\left(\begin{matrix} -(1+\mathrm{a}) \\ 0, (1+\mathrm{b}) \end{matrix} \middle| -\mathrm{x}\right), \tag{18}$$

Putting Equation (15) in Equation (6) and Equation (16) in Equation (11), we get the following indefinite integrals of Meijer's G-function:

$$\int x^{mb-1}e^{-x}G_{1,2}^{1,1}\left(\begin{matrix}-a\\0,-b\end{matrix}\middle|-x\right)^{m-1}G_{1,2}^{1,1}\left(\begin{matrix}1-a\\0,1-b\end{matrix}\middle|-x\right)dx = \frac{x^{mb}e^{-x}G_{1,2}^{1,1}\left(\begin{matrix}1-a\\0,1-b\end{matrix}\middle|-x\right)^{m}}{ma} \tag{19}$$

$$\int x(b-x)^{mb-1}G_{1,2}^{1,1}\left(\begin{matrix}1-a\\0,1-b\end{matrix}\middle|-x\right)^{m-1}G_{1,2}^{1,1}\left(\begin{matrix}-(1+a)\\0,-(1+b)\end{matrix}\middle|-x\right)dx = \frac{(b-x)^{mb}G_{1,2}^{1,1}\left(\begin{matrix}1-a\\0,1-b\end{matrix}\middle|-x\right)^{m}}{m} \tag{20}$$

4. Indefinite Integration of Meijer's G-Function when p=2 and q=2

We have derived indefinite integration of Meijer's G–function for the case $p = 2, q = 2$

$$W_2(x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}G_{2,2}^{1,1}\left(\begin{matrix}1-a,1-b\\0,1-c\end{matrix}\middle|-x\right), \tag{21}$$

The first and second derivatives of equation are:

$$W'_2(x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}G_{2,2}^{1,1}\left(\begin{matrix}-a,-b\\0,-c\end{matrix}\middle|-x\right), \tag{22}$$

$$W''_2(x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}G_{2,2}^{1,1}\left(\begin{matrix}-(1+a),-(1+b)\\0,-(1+c)\end{matrix}\middle|-x\right). \tag{23}$$

Which satisfies the differential equation:

$$W''_2(x) + \frac{c-x(a+b+1)}{x(1-x)}W'_2(x) - \frac{ab}{x(1-x)}\,W_1(x) = 0. \tag{24}$$

For which the integrating factors are:

$$g(x) = x^{c}(1-x)^{a+b-c+1}, \tag{25}$$

$$g_1(x) = (c-(a+b+1)x)^{\frac{ab}{a+b+1}}, \tag{26}$$

putting the values of Equation (24) in Equation (6) and Equation (25) in Equation (11), one gets:

$$\int x^{mc-1}(1-x)^{m(a+b-c+1)-1}$$

$$G_{2,2}^{1,1}\begin{pmatrix}-a,-b \\ 0,-c\end{pmatrix}|-x\Big)^{m-1} G_{2,2}^{1,1}\begin{pmatrix}1-a,1-b \\ 0,1-c\end{pmatrix}|-x\Big)\,dx \tag{27}$$

$$=\frac{x^{mc}(1-x)^{m(a+b-c+1)}G_{2,2}^{1,1}\left(\begin{smallmatrix}1-a,1-b \\ 0,1-c\end{smallmatrix}|-x\right),^{m-1}}{mab}$$

$$\int x(1-x)(c-(a+b+1)x)^{\frac{a+b+1}{mab}-1}$$

$$G_{2,2}^{1,1}\begin{pmatrix}1-a,1-b \\ 0,1-c\end{pmatrix}|-x\Big)^{m-1} G_{2,2}^{1,1}\begin{pmatrix}-(1+a),-(1+b) \\ 0,-(1+c)\end{pmatrix}|-x\Big)\,dx \tag{28}$$

$$=\frac{(c-(a+b+1)x)^{\frac{a+b+1}{mab}}G_{2,2}^{1,1}\left(\begin{smallmatrix}1-a,1-b \\ 0,1-c\end{smallmatrix}|-x\right)^{m}}{m}.$$

CONCLUSION AND FURTHER WORK

Conway (2015) evaluated the integration of confluent and Gauss hypergeometric functions but both are the particular cases of Meijer's G-function. We have evaluated the integration of Meijer's G-function. We have also obtained integration of some additional relations and recurrence relations of Meijer's G-function and Euler identity. In our work, we have used Meijer's-G function for parameter values $p=1, q=2$ and $p=2, q=2$, which satisfy the second-order linear differential equation. The open problem is that if one can extend the order of differential equation, then for parameter $p>2, q>2$ cases may be defined.

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EVALUATION OF ONLINE EDUCATION PLATFORMS DUE TO COVID-19 BY USING INTERVAL-VALUED Q-RUNG ORTHOPAIR FUZZY MACLAURIN SYMMETRIC MEAN OPERATORS

Mahnoor Nawaz, Madiha Ghamkhar[§], Saima Noreen and **Ghulam Farid**
Department of Mathematics and Statistics, University of Agriculture
Faisalabad, Pakistan.
[§]Email: madiha.ghamkhar@uaf.edu.pk

ABSTRACT

Many academic organizations have adopted an online learning system at the time of the COVID-19 epidemic. Virtual learning systems can facilitate distant learning and ensure that educational resources are distributed fairly. One of the newest communication and information technologies is online learning. Various education networks have varying levels of performance and quality. The Maclaurin symmetric mean operator is a helpful for combining data. It is capable of capturing the interconnections between several source parameters. The online learning systems will be evaluated by using the interval-valued q-rung orthopair fuzzy weighted Maclaurin symmetric mean operator and interval-valued q-rung orthopair fuzzy weighted dual Maclaurin symmetric mean operator. A comparative study will also be presented in this work.

KEYWORDS

Multiple attribute group decision making, interval-valued q-rung orthopair fuzzy weighted maclaurin symmetric mean ($IV\,q-ROFWMSM$) operator, interval-valued q-rung orthopair fuzzy weighted dual maclaurin symmetric mean ($IV\,q-ROFWDMSM$) operator.

1. INTRODUCTION

Online learning is a form of learning that is given over the internet. Students who are unable to participate in on campus classes might benefit from online learning. By government mandate, all educational institutions were shuttered to stop the spread of Covid-19. Now, one of the newest communication and information technologies is online learning (Senol, Lesinger and Çaglar, 2021).

Probability and statistics concepts have long been the key ideas and methods for modelling real-world ambiguities and one of these concepts is fuzzy set idea, which is a variant of classical or crisp mathematics. In set theory, a component can be a part of a set or not (Zimmermann, 2010). The rising complications of the structure make it harder for policymakers to identify the best option from a group of possibilities. Human reasoning is not necessarily based on dual concept; it might be ambiguous, inaccurate, or hazy. Crisp set becomes ineffectual in the face of this ambiguity. Fuzzy set looks to be more successful in producing more exact and appropriate outcomes. Zadeh (1965) was the first to propose the fuzzy set concept in 1965. The membership level of a fuzzy set ranges from [0,1].

Artificial intelligence, computer programming, operational research and robotics are just a few of the domains where fuzzy set theory is used (Zimmermann, 2010).

Atanassov (1986 & 1989) developed the idea of non-membership level ranges from [0,1] in fuzzy sets, as well as intuitionistic fuzzy sets wherein average of membership and non-membership levels is below or equivalent to 1. A generalization of FS is the IFS. Yager offered the PFS. PFS has a quadratic summation of membership and non-membership levels that is below or equivalent to 1. The Pythagorean fuzzy set can fix issues that IFS can't even solve. The quadratic summation of membership and non-membership levels is found to be more than 1 in some circumstances. Based on intutionistic and Pythagorean fuzzy sets, Yager (2016) developed the q-ROFS, where the summation of qth power of membership level and non-membership level is below or equivalent to 1. IFS and PFS are shown to be particular instances of q-ROFS. It's worth noting that the number of permissible orthopairs rises as the rung progresses.

Many decision-making challenges make it tough for policy makers to deliver their thoughts in a precise manner. Wang et al. (2019) have developed a novel idea of IV-qROFSs, wherein the level of membership and non-membership is determined through interval value. It's worth noting that if the top and bottom interval limitations are the same, the IVq-OFSs become q-ROFSs (Ju et al., 2019).

Various accumulation techniques have been established in fuzzy theory for the accumulation of data in an unstable environment. These aggregation techniques have the capacity to aggregate enormous amounts of individual data and evaluate the options based on their overall worth. Maclaurin created the Maclaurin symmetric mean operator that is capable to represent the correlation between several source parameters. The MSM operator may give more powerful information fusion, making it more suitable for solving MAGDM with independent characteristics (Dong and Geng, 2021). The remaining paper is organized as follows. Section 2 contains definitions of IVq-ROFSs and MSM operator. Section 3 recalls IVq-ROFWMSM and IVq-ROFWDMSM operators. Section 4 presents evaluation of online education platform performance. Section 5 summarizes the study.

2. PRELIMINARIES

2.1 Interval-Valued q-Rung Orthopair Fuzzy Set

We will present the following definition of the IVq-ROFSs based on IVPFSs (Li, Wei, and Lu (2018)) and q-ROFSs (Yager, 2016).

Definition 1 (Wang et al., 2019). Assume U is a set. A IVq-ROFS is an object of the form

$$\tilde{P} = \{\langle u, (\tilde{\theta}_{\tilde{P}}\ (u), \tilde{\theta}_{\tilde{P}}(u))\rangle | u \in U\}, \tag{1}$$

where $\tilde{\theta}_{\tilde{P}}(u) \subset [0,1]$ and $\tilde{\vartheta}_{\tilde{P}}\ (u) \subset [0,1]$ are interval numbers, $\tilde{\theta}_{\tilde{P}}(u) = [\imath(u),\kappa(u)]$, $\tilde{\vartheta}_{\tilde{P}}\ (u) = [\sigma(u),\varsigma(u)]$ with the condition $0 \le (\kappa(u))^q + (\varsigma(u))^q \le 1$, $\forall$u$\in U$,$q \ge 1$. Let = ([$\imath,\kappa$],[σ,ς]) be a q-RIVOFN, $R(\tilde{p}) = 1/4[(1 + \imath^q + \kappa^q) + (1 - \sigma^q - \varsigma^q)]$ and $S(\tilde{p}) = (\imath^q + \kappa^q + \sigma^q - \varsigma^q)/2$ are the score and accuracy functions of a IVq-ROFN $\tilde{p}$.

2.2 The Maclaurin Symmetric Mean Operator

Definition 2 (Maclaurin, 1729). Let $\tilde{p}_j (j = 1,2, \ldots, m)$ be a set of non-negative real numbers, and $n = (1,2, \ldots, m)$. If

$$MSM^{(n)}(\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_m) = \left(\frac{\sum_{1 \le j_1 < j_2 < \cdots < j_n \le m} \prod_{k=1}^{n} \tilde{p}_{j_k}}{C_m^n} \right)^{\frac{1}{n}} \tag{2}$$

$MSM^{(n)}$, where $(j_1, j_2, \ldots, j_n)$ covers all n-tuple combinations of (1,2,...,m), and C_m^n is the binomial coefficient, became known as the MSM operator.

3. SOME MSM OPERATOR WITH IVq-ROFNS

3.1 The IVq-ROFWMSM Operator

Definition 3 (Wang et al., 2019). Let $\tilde{p}_k = ([\iota_k, \kappa_k], [\sigma_k, \varsigma_k])(k=1,2,\ldots,m)$ be a set of IVq-ROFNs and their weight vector be $v_i = (v_1, v_2, \ldots, v_m)^T$, thereby satisfying $v_i \in [0,1]$ and $\sum_{i=1}^{m} v_i = 1$. Then IVq-ROFWMSM operator is defined as

$$IVq - ROFWMSM^{(n)}(\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_m) = \left(\frac{\sum_{1 \le j_1 < \cdots < j_k \le m} \prod_{k=1}^{n} \left(\tilde{p}_{j_k}\right)^{v_{j_k}}}{C_m^n} \right)^{\frac{1}{n}}$$

$$= \left\{ \begin{array}{l} \left[\left(\sqrt[q]{1 - \prod_{1 \le j_1 < j_2 < \cdots < j_n \le m} \left(1 - \left(\prod_{k=1}^{n} (\iota_k)^{v_{j_k}} \right)^q \right)^{\frac{1}{C_m^n}}} \right)^{\frac{1}{n}}, \left(\sqrt[q]{1 - \prod_{1 \le j_1 < j_2 < \cdots < j_n \le m} \left(1 - \left(\prod_{k=1}^{n} (\kappa_k)^{v_{j_k}} \right)^q \right)^{\frac{1}{C_m^n}}} \right)^{\frac{1}{n}} \right], \\ \left[\sqrt[q]{1 - \left(1 - \left(\prod_{1 \le j_1 < j_2 < \cdots < j_n \le m} \left(1 - \prod_{k=1}^{n} \left(1 - (\sigma_k)^q\right)^{v_{j_k}} \right) \right)^{\frac{1}{C_m^n}} \right)^{\frac{1}{n}}}, \sqrt[q]{1 - \left(1 - \left(\prod_{1 \le j_1 < j_2 < \cdots < j_n \le m} \left(1 - \prod_{k=1}^{n} \left(1 - (\varsigma_k)^q\right)^{v_{j_k}} \right) \right)^{\frac{1}{C_m^n}} \right)^{\frac{1}{n}}} \right] \end{array} \right\}. \tag{3}$$

3.2 The IVq-ROFWDMSM Operator

Definition 4 (Wang et al., 2019]. Let $\tilde{p}_j = ([\iota_k, \kappa_k], [\sigma_k, \varsigma_k])(k = 1,2,\ldots,m)$ be a set of $IV\, q - ROFNs$ and their weight vector be $v_j = ((v_1, v_2, \ldots, v_n)^T$ thereby satisfying $v_i \in [0,1]$ and $\sum_{r=1}^{m} v_i = 1$.

$$IVq-ROFWDMSM^{(n)}(\tilde{p}_1,\tilde{p}_2,\dots,\tilde{p}_m)=\frac{1}{n}\left(\prod_{1\le j_1<j_2<\cdots<j_n\le m}\left(\sum_{k=1}^{n}v_{j_k}\tilde{p}_{j_k}\right)^{\frac{1}{C_m^n}}\right)$$

$$=\left\{\begin{array}{l}\left[\sqrt[q]{1-\left(1-\prod_{1\le j_1<j_2<\cdots<j_n\le m}\left(1-\prod_{k=1}^{n}\left(1-(\iota_k)^q\right)^{v_{j_k}}\right)^{\frac{1}{C_m^n}}\right)^{\frac{1}{n}}},\ \sqrt[q]{1-\left(1-\prod_{1\le j_1<j_2<\cdots<j_n\le m}\left(1-\prod_{k=1}^{n}\left(1-(\kappa_k)^q\right)^{v_{j_k}}\right)^{\frac{1}{C_m^n}}\right)^{\frac{1}{n}}}\right]\\ \left[\left(\sqrt[q]{1-\prod_{1\le j_1<j_2<\cdots<j_n\le m}\left(1-\left(\prod_{k=1}^{n}(\sigma_k)^{v_{j_k}}\right)^q\right)^{\frac{1}{C_m^n}}}\right)^{\frac{1}{n}},\ \left(\sqrt[q]{1-\prod_{1\le j_1<j_2<\cdots<j_n\le m}\left(1-\left(\prod_{k=1}^{n}(\varsigma_k)^{v_{j_k}}\right)^q\right)^{\frac{1}{C_m^n}}}\right)^{\frac{1}{n}}\right]\end{array}\right\}. \tag{4}$$

4. EVALUATION OF ONLINE EDUCATION PLATFORMS PERFORMANCE

4.1 Numerical Example

Technology advancements have opened up new development options for the education business. Mobile Internet and 5G communication technology have boosted the growth of online education. COVID-19 is affecting a growing number of schools and educational institutions in India. Institutes are implementing online teaching strategies in order to enhance student progress. As the number of online education platforms grows, the public's attention has been attracted to the difference in quality among them. As a result, assessing the effectiveness of online learning systems is critical.

4.2 Evaluation Process of Online Education Platforms Performance

Assume that three decision-makers (D_1, D_2, D_3) are invited to evaluate the performance of four online education platforms, which can be denoted as G_1, G_2, G_3 and G_4. Weight vector of decision maker is $v = (0.3, 0.3, 0.4)^T$ Decision-makers evaluate the performance of the four education platforms under four attributes, i.e., availability (A_1), interactive quality (A_2), system quality (A_3), and quality of service (A_4). Weight vector of attributes is $v = (0.40, 0.25, 0.20, 0.15)^T$ (Wang and Zhou, 2021).

Step 1: To obtain the entire IVq-ROFNs p^{α}_{jk}(j = 1,2,3,4,k = 1,2,3,4) of G_j, combine all IVq-ROFNs p^{α}_{jk} using the IVq-ROFWA(IVq-ROFWG) operator. Table 4 (with q = 4) displays the results.

Table 1
IVq-ROFN Decision Matrix (E_1)

	A_1	A_2	A_3	A_4
G_1	([0.7,0.8],[0.5,0.6])	([0.6,0.8],[0.4,0.7])	([0.7,0.9],[0.3,0.6])	([0.6,0.7],[0.4,0.5])
G_2	([0.7,0.9],[0.4,0.6])	([0.6,0.9],[0.2,0.4])	([0.8,0.9],[0.2,0.3])	([0.6,0.8],[0.2,0.5])
G_3	([0.5,0.7],[0.1,0.2])	([0.6,0.8],[0.1,0.3])	([0.6,0.7],[0.3,0.4])	([0.7,0.8],[0.4,0.5])
G_4	([0.7,0.9],[0.4,0.5])	([0.6,0.8],[0.3,0.5])	([0.5,0.6],[0.1,0.2])	([0.7,0.9],[0.3,0.5])

Table 2
IVq-ROFN Decision Matrix (E_2)

	A_1	A_2	A_3	A_4
G_1	([0.6,0.7],[0.1,0.3])	([0.8,0.9],[0.3,0.4])	([0.6,0.9],[0.2,0.4])	([0.6,0.7],[0.3,0.6])
G_2	([0.8,0.9],[0.2,0.4])	([0.7,0.8],[0.1,0.3])	([0.6,0.7],[0.1,0.2])	([0.7,0.9],[0.5,0.6])
G_3	([0.8,0.9],[0.2,0.4])	([0.7,0.9],[0.2,0.3])	([0.5,0.8],[0.3,0.5])	([0.6,0.8],[0.3,0.4])
G_4	([0.6,0.8],[0.2,0.3])	([0.7,0.9],[0.6,0.7])	([0.6,0.7],[0.3,0.4])	([0.3,0.5],[0.1,0.2])

Table 3
IVq-ROFN Decision Matrix (E_3)

	A_1	A_2	A_3	A_4
G_1	([0.5,0.7],[0.3,0.4])	([0.5,0.6],[0.3,0.5])	([0.2,0.4],[0.1,0.2])	([0.2,0.4],[0.2,0.4])
G_2	([0.5,0.7],[0.3,0.4])	([0.4,0.5],[0.2,0.3])	([0.5,0.7],[0.1,0.2])	([0.4,0.5],[0.2,0.4])
G_3	([0.1,0.2],[0.2,0.4])	([0.7,0.8],[0.6,0.8])	([0.6,0.8],[0.5,0.8])	([0.1,0.2],[0.4,0.5])
G_4	([0.2,0.3],[0.5,0.7])	([0.3,0.4],[0.2,0.5])	([0.4,0.5],[0.5,0.6])	([0.6,0.8],[0.6,0.8])

Table 4
The Aggregated Results by the *IV q − ROFWA* Operator

	A_1	A_2
G_1	([0.6108,0.7374],[0.2515,0.4143])	([0.6704,0.8020],[0.3270,0.5172])
G_2	([0.6954,0.8524],[0.2895,0.4517])	([0.5956,0.7993],[0.1624,0.3270])
G_3	(([0.6352,0.7590],[0.1624,0.3249])	([0.6758,0.8401],[0.2520,0.4441])
G_4	([0.5852,0.7863],[0.3552,0.4907])	(([0.5880,0.7886],[0.3140,0.5531])
	A_3	A_4
G_1	([0.5852,0.8317],[0.1711,0.3423])	([0.5327,0.6332],[0.2780,0.4830])
G_2	([0.6704,0.7958],[0.1231,0.2258])	([0.5956,0.7933],[0.2632,0.4830])
G_3	([0.5757,0.7763],[0.3680,0.5643])	([0.5846,0.7220],[0.3669,0.4676])
G_4	([0.5127,0.6108],[0.2646,0.3821])	([0.6017,0.8066],[0.2847,0.4583])

Table 5
The Aggregated Values by the *IV q – ROFWMSM* (*IV q − ROFWDMSM*) Operator

	IV q – ROFWMSM	*IV q − ROFWDMSM*
G_1	([0.8861,0.9615],[0.1841,0.3083])	([0.4465,0.5722],[0.7438,0.8284])
G_2	([0.8994,0.9511],[0.1596,0.2712])	([0.4080,0.5555],[0.7566,0.8334])
G_3	([0.8919,0.9409],[0.2247,0.3352])	([0.4639,0.6024],[0.7016,0.7947])
G_4	([0.8742,0.9331],[0.2141,0.3382])	([0.4325,0.5601],[0.7297,0.8209])

Step 2: The fused values for k=2 are given in Table 5 and are obtained by fusing all IV q − ROFNs together by IV q − ROFWMSM (IV q −ROFWDMSM).

Step 3: The scores of the alternatives are displayed in Table 6 based on the fused values in Table 5 and the scoring functions of IVq-ROFNs.

Step 4: Use the values from Table 6 to rank all of the choices, and the results are given in Table 7. G3 is, without a doubt, the best option.

Table 6
The Score Values of Alternatives

	$IV\,q - ROFWMSM$	$IV\,q - ROFWDMSM$
G_1	0.8503	0.3425
G_2	0.8317	0.3282
G_3	0.8666	0.3842
G_4	0.8652	0.3496

Table 7
Ordering of Alternatives

	Ordering
IVq−ROFWMSM operator	$G3 > G4 > G1 > G2$
IVq−ROFWDMSM operator	$G3 > G4 > G1 > G2$

4.3 Comparative Analysis

The IVq-ROFWMSM and IVq-ROFWDMSM operators are now compared to the IVq-ROFWA and IVq-ROFWG operators. Table 8 displays the results.

Table 8
Comparative Results

	Ordering
$IV\,q - ROFWA$ operator	$G3 > G1 > G4 > G2$
IV q-ROFWG operator	$G3 > G4 > G1 > G2$

CONCLUSION

The MSM operator can handle MAGDM problems with IVq-ROF information and can take into account the interrelationships between arguments. Based on IVq-ROFWMSM and IVq-ROFWDMSM operators, we evaluate the online education platform performance problem, which helps the decision makers to choose the best platform. In the future, we will investigate more MAGDM methods under IV-qROFSs situations and study further real-life applications in many other fields under uncertain environment.

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COMPARISON OF AUTOREGRESSIVE INTEGRATED MOVING AVERAGE AND ARTIFICIAL NEURAL NETWORK FOR WEATHER FORECASTING IN PAKISTAN

Muhammad Waqar Ashraf[1], Muhammad Imran Khan[1], Maqbool Ahmad[1], Muhammad Waqas Ashraf[2], Muhammad Wahab Ashraf[3] and **Sidra Nosheen[4]**

[1] Department Mathematics and Statistics, University of Agriculture Faisalabad, Pakistan. Email: waqarashraf429@gmail.com
imran223im@gmail.com
maqboolahmadstd2@gmail.com

[2] Electrical Engineering Department, University of Engineering & Technology, Lahore, (Faisalabad Campus), Pakistan
Email: drwaqasashraf2345@gmail.com

[3] Department of Chemistry, Government College University Faisalabad Faisalabad, Pakistan. Email: wahabashraf004@gmail.com

[4] Department of Mathematics, Government College Women University Faisalabad, Pakistan. Email: sidranosheen199@gmail.com

ABSTRACT

The Auto-regressive Integrated Moving Average (ARIMA) testing technique is a climate forecasting statistical model. ARIMA model trains itself by following the algorithm again and again and then use for forecasting. The Artificial Neural Network (ANN) technique exists as a computational model created on human neural-system. The working of the ANN technique is built on the base of structure & functions of biological neural-system. In this study, first, the model of ARIMA is used in weather estimating in Pakistan and then the ANN model is applied at the missing or error values which did not identify by the ARIMA model. For weather forecasting in Pakistan, ARIMA model has been compared with ANN model. Also, Root Mean Square Error (RMSE) has been used for the association of efficiency of ANN & ARIMA techniques. The best-fit model has been selected according to its forecasting performance. The criterion commonly has been used for the selection of statistical models: Moving Average-Error (MAE), Mean-Error (ME), Mean-Absolute Scaled-Error (MASE) and Root-Mean Square-Error (RMSE) have been compared to select a suitable approach for weather forecasting about Pakistan. The ARIMA model has been found the more appropriate for weather forecasting in Pakistan.

1. INTRODUCTION

Weather conditions predicting (mainly rain-fall forecasting) is individual of the maximum significant & difficult functioning jobs that meteorological agencies throughout the globe do. It's also a time-consuming process that entails many specialist areas of knowledge. Investigators of that area must divided climate estimating methods keen on dual major divisions, which are numerical modelling and scientific analysis of meteorological data, according to the researchers. The most often used approaches for predicting rainfall are numerical and statistical. Despite the fact that research in these areas has been going on aimed at a several period, the success of techniques are seldom apparent. Around is very little effectiveness of predicting climate parameters by means of numerical technique. The precision of these designed statistical methods is determined by intrinsically imperfect starting circumstances. In local and short-term situations, these technologies are unable to provide acceptable outcomes. The performance, however, is inadequate for durable prediction of moon-soon rain-fall, even at a wider geographic scale, and especially in Pakistan's area. Statistical techniques that consider rainfall time series as stochastic are extensively applied for long-term rain-fall prediction. ARIMA has used statistical methods to forecast moon-soon precipitation. In years with average monsoon rainfall, statistical models were effective; but, in decades with severe monsoon rainfall, such as 2002 and 2004, statistical models were a complete failure. But it is actual tough to acquire the equal or greater competence in forecasting district region level moon-soon rain-fall using such statistical methods.

Sanjeev (2008) discovered that there is no other model than ANN had been capable to predict district region durable climate parameters with accuracy in India. Liu *et al.,* (2016) said in their research that a period chain is a set of quantitative measurements occupied at various moments in domain. Hyndman and Athanasopoulos (2018) said that various revisions on GDP development had been utilized various types of forecast techniques to estimate GDP.

Bönner (2009) provided comparison of ANN technique to VAR & ARIMA in predicting the Euro's inflation rate, and came to the conclusion that linear techniques (VAR and ARIMA) remain a subsection of nonlinear techniques like Neural-Networks. Aminian *et al.,* (2006) observed that the lack of a standard approach to assess the comparative predicting benefit of a neural-network technique is a loss to estimate the accuracy of ANN models over linear models in forecast. The most common performance actions utilized to show the prediction superiority of a pattern are the average APE, MSE, RMSE, SSE, MAE and determination factor.

Zhang (2003) discovered that ARIMA techniques are adaptable since they might epitomize a diversity of time-series kinds, together with clean auto-regressive (AR), clean moving-average (MA), and assorted AR and MA (ARMA) chain. The pre-supposed direct shape of the technique is, however, its primary constraint. Granger (1989) also said that ARIMA is a useful tool for forecasting because the Box–Jenkins technique concentrates on little instruction auto-correlation, a method is regarded acceptable if little-order auto-correlations have no importance equally if higher order AC are significant, resulting in inferior models. Darji *et al.,* (2015) introduced that the ANN is made up of three layers: input, output, and one or more hidden layers. Beforehand charting the in-put/out-put coating, the hidden coating can be used to conduct intermediate calculations.

Taskaiya-Teimzel and Casey (2005) said that the capacity of artificial neural-networks to express non-linearity between input and output variables is its major benefit. Zhang (2003) furthermore described that ANN techniques must an improvement above additional non-linear methods in that these are worldwide estimates; essentially non-parameteric techniques that can accurately estimate a wide range of purposes. Darji *et al.,* (2015) discovered that altered ANN back-promulgation preparation methods, such as conjugate-gradient ancestry and Leven-berg Marquardt, be present employed to update the synaptic weights.

2. METHODOLOGY

The study uses an experimental research methodology since the goal is to anticipate for the next three years. The information used is secondary. The Pakistan Meteorological Department and the World Bank's Meta Department provided the data. The chosen curriculums of representations that we drive investigate are ARIMA & ANN techniques. The R Soft wear is used for this study.

2.1 ARIMA Model

Technique take up that interval t is a separate variable, y_t represents the surveillance at time interval t, the weights θ_i and α_i and t represents the zero mean chance sound at interval t. The $ARMA\ (p, q)$ technique is a mixture of $AR\ (p)$ and $MA\ (q)$ methods and its significant supposition is that phase series statistics data are static. This technique takes up that y_t is produced over the procedure:

$$y_t = \sum_{i=1}^{q} \theta_i \mu_{t-i} + \sum_{i=1}^{p} \alpha_i y_{t-i} + \mu_t. \tag{2.1}$$

here t is a zero-mean sound tenure.

Thus, $ARIMA\ (p, d, q)$, allows distinctions of information if the model parameter is set; p, numeral of auto-regressive components, d & q, amount of moving-means, has a good description of the non-stationary data series. The restrictions are eventually swapped by number values to readily identify ARIMA.

2.2 ANN Model

Szkuta *et al.*, (1999) said that transitory discovery, design gratitude, estimate, and time-series forecast are just a few of the applications of artificial neural networks (ANN). The word "ANN" refers to a variety of highly linked basic dispensation components that provide another to traditional calculating methods. Unlike previous approaches, ANN shows connected objects by knowledge from illustration data instead than demonstrating procedures.

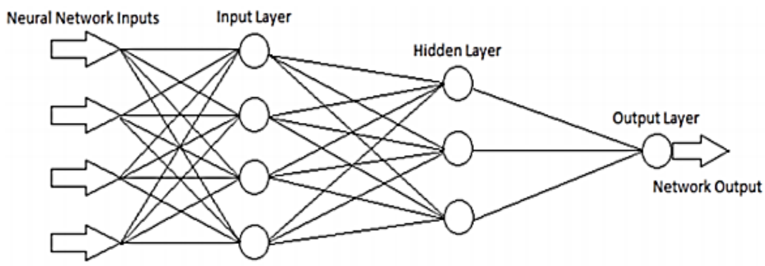

Figure 2.1: Artificial Neural Network Flow Diagram

A network of perceptron is made up of numerous neurons that are all linked together by basic processing units that are similar to one another. Each neuron in the system adds up the biased responses it receives to create an interior action level that can be measured by w_i:

$$w_i = \sum_{k=1}^{n} y_{ik} z_{ik} - u_{i0} \tag{2.2}$$

where y_{ik} is the biased function of the joining beginning response k to neuron i, z_{ik} is the response indicator value k to i, w_{i0} is the thresh-hold related with component i. The output of neuron w_i is

$$w_i = \frac{1}{1 + \exp(-y_i)} \tag{2.3}$$

The input and occlusion layer, the hidden neuron and outputs are changed during training to make the gradient-multiplied by the learner rate restriction.

3. RESULTS AND DISCUSSION

The secondary data gotten from the Pakistan Climatological Branch and the World Data Bank is hourly from 1990 to 2021. The parameters of ARIMA and ANN model for predicting are determined using a rolling development frame consisting of predicting data from the previous three years. The RMSE of a weather forecast is computed to quantify the difference between the estimated value and the true value in instruction to evaluate the forecasting presentation within each forecasting classical. After the first difference in the Model parameters of ARIMA, data becomes stationary. As a result, the ARIMA model's variable d is set to 1. The strategies of ACF and PACF determine the order of p and q. Here's, a variety of ARIMA techniques through p ranging afterwards 0 to 4 and q ranging commencing 0 to 2 are functional to the prediction values. The highest performance for the RMSE of a 1-step prediction was obtained for $ARIMA(1,0,1)(0,1,0)[360]$. For ANN models, Number of concealed neurons/layers and postponements must be changed, and the exercised procedures must be varied multiple times before the validation process yields sufficient accuracy.

Table 3.1
RMSEs of the Predicted Values Intended for Active to 3 Stages Fast of $\boldsymbol{ARIMA(1,0,1)(0,1,0)[360]}$ and ANN Models with Hidden 20 Layers Techniques

Models	**ANN**	**ARIMA**	**ANN**	**ARIMA**	**ANN**	**ARIMA**
Forecast Steps	**One Step**	**One Step**	**Two Step**	**Two Step**	**Three Step**	**Three Step**
Minimum RMSEs	2.7144	2.6443	4.1038	3.9751	5.3986	5.1263
RMSE of PF	2.7357	2.7357	4.1937	4.1937	5.4622	5.4622

Persistence forecasting (PF), the simplest kind of relatively brief prediction, is another technique for making predictions. The RMSEs of PF are too provided in table 4.1 to allow the model to be compared using this easy technique.

ARIMA model always exhibits fewer RMSEs in this investigation than the ANN. Furthermore, when the horizon estimate grows, the $ARIMA\ (1,0,1)(0,1,0)[360]$ model improves farther than the RMSE artificial neural network. In addition, when compared to the RMSEs of PF, both techniques' RMSEs are all lower. In three-step predictions, the ARIMA model outperformed the PF model. In addition, the ANN improves its efficiency while forecasting three steps ahead. As the number of prediction stages increases, the ARIMA model outperforms the artificial neural network, notably in the third phase.

3(a) Calculations and Graphs

The data analysis is start with ANN and the outputs are given in the tables. Also, the graphs indicate the forecasting results.

Table 3.2
Detail about Data which is under Consideration

Values	Min. Temp.	Max. Temp.	Temp.	Visibility	Precip. (mm)	Wind Speed	Wind Direction	Cloud Cover
Min.	32.60	42.90	40.40	0.00	0.00	0.00	50.00	0.00
1st Quartile	53.70	73.50	62.20	2.00	0.00	4.70	145.00	4.30
Median	68.10	88.80	78.50	2.30	0.00	6.90	214.00	22.50
Mean	66.32	85.88	75.56	2.12	0.09	6.99	214.20	28.23
3rd Quartile	80.70	97.80	89.40	2.50	0.00	9.20	283.30	46.30
Max.	89.70	114.00	100.40	5.80	11.89	25.30	360.00	100.00

After training of the model, the results as given below:

Table 3.3
Training of ANN Model using One Hidden Layer

Variables	Values	Variables	Values	Variables	Values
Minimum Temperature	77.10	**Precipitation**	0	**Visibility**	2.32
Maximum Temperature	98.70	**Wind Speed**	11.40	**Cloud Cover**	28.87
Temperature	47.20	**Wind Direction**	248.33	**Relative Humidity**	13.08
Heat Index	17.43	**Visibility**	2.32		

The following figure shows the actual value of the data of weather forecasting:

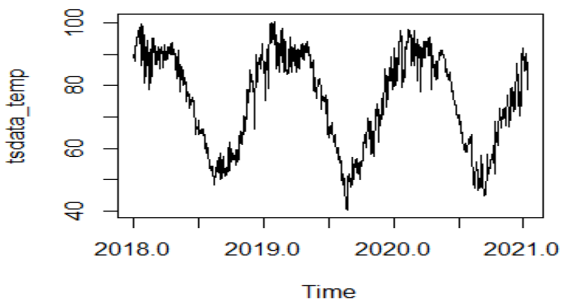

Figure 3.1: **Actual Value of Forecasting Data using ARIMA Model**

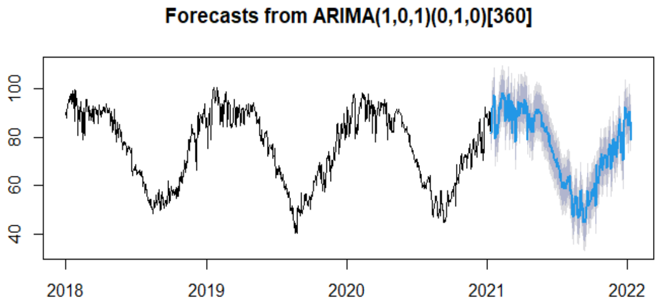

Figure 3.2: Forecast from $ARIMA(1,0,1)(0,1,0)[360]$

From the above, this figure shows the predication for the year 2022 using $ARIMA(1,0,1)(0,1,0)[360]$, which help for the future forecasting for next 10 decays.

The time series is used for the data tsdata_temp over the $ARIMA(1,0,1)(0,1,0)[360]$ model using co-efficient AR(1) and MA(1) with the values 0.7295 and 0.1652 respectively and the standard error of both AR(1) and MA(1) are 0.0320 and 0.0475 respectively.

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EXPECTED BAYESIAN ESTIMATION OF RELIABILITY OF SIZE BIASED BINOMIAL DISTRIBUTION AND ITS EXPECTED POSTERIOR RISK UNDER DIFFERENT LOSS

Madiha, Muhammad Zafar Iqbal, Madiha Ghamkhar
and **Adeel Anjum**
Department of Mathematics and Statistics
University of Agriculture Faisalabad. Pakistan
Email: mfg7023@gmail.com
mzts2004@uaf.edu.pk
madiha_gm@yahoo.com
adeelanjum1954@gmail.com

ABSTRACT

Size Biased Binomial Distribution (SBBD) is used to investigate reliability of E-Bayesian Estimation (EBE) and their E-Posterior Risk (EPR) by using various Loss Functions (LF), Square Error Loss Function (SELF), Weighted Square Error Loss Function (WSELF), Precautionary Loss Function (PLF), K-Loss Function (KLF). Power distribution (PD) and Uniform distribution (UD) used as a prior. Based on EBE E-Posterior Risk (EPR) will be established. Monte-Carlo (MC) Technique used to study relationship between EB and EBE. The comprehensive simulation study used to compare the performance of Estimation Methods by using R software.

1. INTRODUCTION

Bayesian estimation could be a recent parameter estimation approach utilized in applied math .The BE expectation regarding hyper parameter of distribution consist of EBE. Bayesian methodology to applied math enable user to create changes to their model looking on previous proof and information. They are presently applied in a very massive fields of science and together with medical analysis and medicine, scientific discipline, and type of alternative fields. Suppose that Y is number of failures in n independent trials, R be probability of success with a Size Biased Binomial Distribution (SBBD) is defined as,

$$P(Y=r) = {}^{n-1}C_{r-1} R^{n-r-1}(1-R)^{r}, \ 0<R<1, \ r=0,1,2,...,n. \tag{1}$$

Han (2007) recommended EB estimating procedure to predict rate of failure. Formulas for EBE of rate of failure had projected, also because of EBE property. EB and EBE are easy to use. Mir (2008) firstly introduced the moments of a Size Biased Generalized negative Binomial Distribution. A beta distribution provided previous data on one parameter. To estimate Thomas Bias for a constant quantity one parameter has been used. Han (2011) created estimation of extra parameter approach referred to as EBE technique to assess reliability of Binomial Distribution. The definition of EB reliableness estimation is projected, likewise as formulas for EBE reliableness and Hierarical Bayesian reliableness. Finally, a numerical example was incontestable. Touw, (2009) study that BE overcomes drawback of drastically varied parameter estimates fitting knowledge equally well only by discovered knowledge to update previous distribution on the parameters. To reduce Bayesian may be specialize in subjective likelihood, taking into consideration a previous predictions. For quantity of arithmetic consideration need integrates work on BE single Weibull Distribution (WD) with work on BE of mixed exponentials. Reyad and Ahmed (2016) planned supported kind II censored ways of Kumaraswamy Distribution for parameter of BE and EBE. To check EBE, MC simulation has been used. The loss functions has been used to generate these estimators. Okasha and Wang (2016) introduced BE and EBE on record statistics for estimating parameter and relibleness of Exponential Distribution. The MC simulation and comparisons are used for computing EBE and BE. .Okasha *et al.*, (2021) introduced BE may be a loss operator that minimizes posterior expected value. EBE may be a BE obtained by calculative mean of BE. The failure rate is calculable using EBE methodology which are special cases of rate parameter, failure rate and quantile functions. The Size Biased Distribution (SBD) may be a form of Weighted Distribution (WD). WD is square measure wont to facilitate in selecting acceptable models for information. It has been employed in a large vary of relibleness applications. The distribution tends to emerge at the purpose wherever length of SBD units of concern is set by weight function and long biased distribution units of concern are said to be disputed. (Al-Kadin and Hussein, 2014).In existing analysis, we can see, compared with BE, EBE is thus straightforward that is easier to perform.

2. DEFINITION OF EBE AND ITS EPR

In this part, definition of EBE and EPR introduced.

2.1 Definition of EBE of R

The original definition of EBE was given by Han (2011).

Definition 1

The EBE is defined as,

$$R_{EB} = E\left[R_B(b)\right] = \int_D R_B(b)\pi(b)db$$

is called Expected Bayesian Estimation (EBE) of R. Where D is domain of b, $R_B(b)$ is BE and $\pi(b)$ is density function with hyper parameter b over domain D.

2.2 Definition of EPR of R

In EBE the Posterior Risk (PR) is used to measure the estimated risk. Ali Aslam and Kazmi (2013) to measure risk of BE used PR.

Definition 2

The Expected Posterior Risk (EPR) is defined as,

$$R(R_{EB}) = E\left[R[R_B(b)]\right] = \int_D R[R_B(b)]\pi(b)db$$

where R_{EB} is the EBE, $R(R_{EB})$ is the EPR of the R_{EB}, D is domain of hyper parameter b and $\pi(b)$ is the density function with hyper parameter b over the domain D.

3. BAYESIAN ESTIMATION(BE) AND ITS POSTERIOR RISK(PR) OF R

Ali. Aslam and Kazmi (2013) introduced BE and its PR for any Prior Distribution (PD), then we have following theorem 1.

Theorem 1

By using different LF, the BE and its PR for any PD of parameter ϑ, we have the following results.

i. If Square Error Loss Function (SELF) is used $L_1(\vartheta,\delta)=(\vartheta-\delta)^2$ then BE is, $E(L_1(\vartheta,\delta)\mid y)=E(\vartheta^2\mid y)+\delta^2-2\delta E(\vartheta\mid y)$

   We can obtain BE of ϑ is, $\vartheta_{B1}(y)=E(\vartheta\mid y)$.

   The Bayesian Posterior Risk (BPR) is actually $R(\delta\mid y)=E(L_1(\vartheta,\delta)\mid y)$ which takes minimum value. Then PR of BE $\vartheta_{B1}(y)$ is $R[\vartheta_{B1}(y)]=\mathrm{var}(\vartheta\mid y)$

ii. If Weighted Square Error Loss Function (WSELF) is used $L_2(\vartheta,\delta)=\frac{(\vartheta-\delta)^2}{\vartheta}$ then BE is, $E(L_2(\vartheta,\delta)\mid y)=E(\vartheta\mid y)+\delta^2E(\vartheta^{-1}\mid y)-2\delta$

   We can obtain BE of ϑ is, $\vartheta_{B2}(y)=[E(\vartheta^{-1}\mid y)]^{-1}$

   The BPR is actually the $R(\delta\mid y)=E(L_2(\vartheta,\delta)\mid y)$ .Which takes minimum value. Then PR of BE $\vartheta_{B2}(y)$ is, $R[\vartheta_{B2}(y)]=E(\vartheta\mid y)-[E(\vartheta^{-1}\mid y)]^{-1}$

iii. If Precautionary Error Loss Function (PLF) is used $L_3(\vartheta,\delta)=\frac{(\vartheta-\delta)^2}{\delta}$ then BE is,

$$E(L_3(\vartheta,\delta)\mid y)=\frac{E\left(\vartheta^2\mid y\right)+\delta^2-2\delta E\left(\vartheta\mid y\right)}{\delta}$$

   We can obtain the BE of ϑ is, $\vartheta_{B3}(y)=\sqrt{E(\vartheta^2\mid y)}$

   The BPR is actually the $R(\delta\mid y)=E(L_3(\vartheta,\delta)\mid y)$ .Which takes minimum value.

   Then PR of BE $\vartheta_{B3}(y)$ is, $R[\vartheta_{B3}(y)]=2\left[\sqrt{E(\vartheta^2\mid y)}-E(\vartheta\mid y)\right]$

iv. If K loss Function (KLF) is used $L_4(\vartheta,\delta)=\left(\sqrt{\frac{\vartheta}{\delta}}-\sqrt{\frac{\delta}{\vartheta}}\right)^2$ then BE is,

$$E(L_4(\vartheta,\delta)\mid y)=\frac{E(\vartheta\mid y)}{\delta}+\delta E(\vartheta^{-1}\mid y)-2$$

We can obtain the BE of ϑ is, $\vartheta_{B4}(y) = \sqrt{\dfrac{E(\vartheta \mid y)}{E(\vartheta^{-1} \mid y)}}$.

The BPR is actually $R(\delta \mid y) = E(L_4(\vartheta, \delta) \mid y)$. Which takes minimum value Then PR of BE $\vartheta_{B4}(y)$ is, $R[\vartheta_{B4}(y)] = 2\left[\sqrt{E(\vartheta^{-1} \mid y)E(\vartheta \mid y)} - 1\right]$.

4. BAYESIAN ESTIMATION (BE) AND EXPECTED BAYESIAN ESTIMATION (EBE) OF R

Theorem 2

For SBBD (1), if Power Distribution(PD) is used as a PD for BE then pdf $\pi(b)$ is given by (2), and Uniform Distribution(UD) is used as a prior distribution for EBE then pdf $\pi(b)$ is given by (3),

$$\pi(R \mid b) = (b+1)R^b , \tag{2}$$

where 'b' is a hyper parameter and $b > 0$.

$$\pi(b) = 1, \ 0 < b < 1 \tag{3}$$

where 'b' is hyper parameter. Then we have following results

i. If SELF is used then BE of R is, $R_{B1}(b) = \dfrac{n+b-r}{n+b+1}$ and the EBE of R is, $R_{EB1}(b) = 1-(1+r)\left[\ln\left(\dfrac{n+2}{n+1}\right)\right]$

ii. If WSELF is used then BE of R is, $R_{B2}(b) = \dfrac{n+b}{n+b-r-1}$ and the EBE of R is, $R_{EB2}(b) = 1-(1+r)\left[\ln\left(1+\dfrac{1}{n}\right)\right]$

iii. If PLF is used then BE of R is, $R_{B3}(b) = \sqrt{\dfrac{(n+b-r)(n+b-r+1)}{(n+b+1)(n+b+2)}}$ and the EBE of R is, $R_{EB3}(b) = \int_0^1 \sqrt{\dfrac{(n+b-r)(n+b-r+1)}{(n+b+1)(n+b+2)}}\,db$

iv. If KLF is used then BE of R is, $R_{B4}(b) = \sqrt{\dfrac{(n+b-r)(n+b-r-1)}{(n+b+1)(n+b)}}$ and the EBE of R is, $R_{EB4}(b) = \int_0^1 \sqrt{\dfrac{(n+b-r)(n+b-r-1)}{(n+b+1)(n+b)}}\,db$.

Proof:

For SBBD (1), the likelihood function is given by $L(r \mid R) = {}^{n-1}C_{r-1} R^{n-r-1}(1-R)^{r}$, $0 < R < 1$

The pdf of the PD $\pi(R \mid b)$ of R is given by (2), then the Posterior Density Function (PDF) of R is defined as,

$$h(R|b) = \frac{\pi(R \mid b) L(r \mid R)}{\int_D \pi(R|b) L(r|R) dR} = \frac{{}^{n-1}C_{r-1} R^{n-r-1}(1-R)^{r}(b+1)R^{b}}{\int_0^1 {}^{n-1}C_{r-1} R^{n-r-1}(1-R)^{r}(b+1)R^{b} dR}$$

$$= \frac{R^{n+b-r-1}(1-R)^{r}}{B(n+b-r, r+1)}$$

i. If SELF is used then by using I of Theorem 1, then BE of R is,

$$R_{B1}(b) = E(R \mid y) = \int_0^1 R h(R \mid b) dR = \frac{n+b-r}{n+b+1}$$

By using definition 1, EBE will be, $R_{EB1}(b) = \int_0^1 R_{B1}(b)\pi(b) db$

$$R_{EB1}(b) = 1-(1+r)\left[\ln\left(\frac{n+2}{n+1}\right)\right] \text{ Since, } \pi(b) = 1.$$

5. POSTERIOR RISK (PR) AND EXPECTED POSTERIOR RISK (EPR) OF R

Theorem 3

For SBBD (1), if the PD is used as a prior distribution for BE and PR then PDF $\pi(b)$ is given by (2), and UD is used as a prior distribution for EBE and EPR then PDF $\pi(b)$ is given by (3), then we have the following conclusion.

i. If SELF is used, then PR of BE $R_{B1}(b)$ is, $R\left[R_{B1}(b)\right] = \frac{(r+1)(n+b-r)}{(n+b+1)^2(n+b+2)}$

and the EPR of R is, $R\left[R_{EB1}(b)\right] = (r+1)\int_0^1 \frac{n+b-r}{(n+b+1)^2(n+b+2)} db$

ii. If WSELF is used, then PR of BE $R_{B2}(b)$ is, $R\left[R_{B2}(b)\right] = \frac{r+1}{(n+b)(n+b+1)}$

and EPR of R is, $R\left[R_{EB2}(b)\right] = (r+1)\int_0^1 \frac{1}{(n+b)(n+b+1)} db$

iii. If PLF is used, then PR of BE $R_{B3}(b)$ is, $R\left[R_{B3}(b)\right]=2\sqrt{\dfrac{(r+1)(n+b-r)}{(n+b+1)^2(n+b+2)}}$

and EPR of R is, $R\left[R_{EB3}(b)\right]=2\sqrt{(r+1)}\int_0^1\sqrt{\dfrac{(n+b-r)}{(n+b+1)^2(n+b+2)}}db$

iv. If KLF is used, then PR of BE $R_{B4}(b)$ is,

$$R\left[R_{B4}(b)\right]=2\left[\sqrt{\frac{(n+b)(n+b-r)}{(n+b+1)(n+b-r-1)}}-1\right]$$

and the EPR of R is, $R\left[R_{EB4}(b)\right]=2\int_0^1\left[\sqrt{\dfrac{(n+b)(n+b-r)}{(n+b+1)(n+b-r-1)}}-1\right]db$.

Proof:

i. From part i and iii of theorem 2, we have the following results

$$E(R\mid y)=\frac{n+b-r}{n+b+1} \text{ and } E(R^2\mid y)=\frac{(n+b-r)(n+b-r+1)}{(n+b+1)(n+b+2)}$$

$$\operatorname{var}(R\mid y)=E(R^2\mid y)-\left[E(R\mid y)\right]^2$$

$$=\frac{(n+b-r)(n+b-r+1)}{(n+b+1)(n+b+2)}-\left[\frac{n+b-r}{n+b+1}\right]^2=\frac{(r+1)(n+b-r)}{(n+b+1)^2(n+b+2)}$$

If SELF is used, then by using part I of theorem 1 the PR of BE $R_{B1}(b)$ is,

$$R\left[R_{B1}(b)\right]=\operatorname{var}(R\mid y)=\frac{(r+1)(n+b-r)}{(n+b+1)^2(n+b+2)}$$

If the pdf $\pi(b)$ of hyper parameter b is given by (3), then by using definition 2 EPR of R is,

$$R\left[R_{EB1}(b)\right]=\int_0^1 R\left[R_{B1}(b)\right]\pi(b)db=(r+1)\int_0^1\frac{n+b-r}{(n+b+1)^2(n+b+2)}db\ .$$

6. MONTE CARLO SIMULATION

In simulation, a sample of randomized data is generated in a way that resembles a real situation, and the sample is summarized in the same way. MC simulation is versatile and can produce a variety of helpful results when used to simulate data. MC simulation is utilized in this section to analyze different loss functions. R software is used for analysis. Under SELF, we take n , b the hyper parameter and r be the shape parameter. We fix value of $b=1$ and random values of $r=1,1.5,2$ and $n=20,60,100,300$. By increasing sample size n our MSE minimizes and Bias also minimizes.

r	n	MSE	Bias
1	20	0.0082	-0.0909
	60	0.0010	-0.0322
	100	0.0038	-0.01963
	300	0.00403	-0.0066
1.5	20	0.3765	-0.61363
	60	0.2919	-0.5403
	100	0.27510	-0.52450
	300	0.25834	-0.50827
2	20	1.2913	-1.1363
	60	1.00991	-1.0483
	100	1.0596	-1.0294
	300	1.0194	-1.009

Under PLF, we take n, b the hyper parameter and r be the shape parameter. We fix value of $b=1$ and different values of $r=1,1.5,2$ and $n=20,60,100,300$. By increasing sample size n our MSE minimizes and Bias also minimizes.

r	n	MSE	Bias
1	20	0.0079	-0.0889
	60	0.00010	-0.03200
	100	0.000380	-0.01951
	300	0.0000437	-0.000661
1.5	20	0.3735	0.6111
	60	0.2916	-0.54000
	100	0.2749	-0.52430
	300	0.2583	-0.5082
2	20	1.2846	-1.13340
	60	1.0983	-1.0480
	100	1.0593	-1.0292
	300	1.0199	-1.0099

Under KLF, we take n, b the hyper parameter and r be the shape parameter. We fix value of $b=1$ and random values of $r=1,1.5,2$ and $n=20,60,100,300$. By increasing sample size n our MSE minimizes and Bias also minimize

r	n	MSE	Bias
1	20	0.00866	-0.0930
	60	0.00107	-0.0325
	100	0.000386	-0.019704
	300	0.000044	-0.0066
1.5	20	0.37988	-0.61634
	60	0.29230	-0.54063
	100	0.27523	-0.52463
	300	0.25830	-0.5082
2	20	1.29072	-1.139617
	60	1.0999	-1.0487
	100	1.05998	-1.02952
	300	1.02	-1.009

CONCLUSION

In this paper studies the EBE and EPR of reliableness under different loss functions were derived under condition of SBBD. Under simulation study we have the following result that $PLF < SELF < KLF$.

Form this paper we conclude that EBE is easy to operate as compare to BE. We also see properties of EBE in the future.

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EXAMINING THE ROLE OF INSTITUTIONAL QUALITY IN ACCELERATING ENTREPRENEURIAL ACTIVITY

Noman Arshed, Nadia Hanif, Abida Hafeez and **Sehar Latif Rana**
Division of Management and Administrative Science,
University of Education, Lahore, Pakistan.
Email: noman.arshed@ue.edu.pk
nadiahanif77@yahoo.com
nadia.hanif@ue.edu.pk
abida.hafeez@ue.edu.pk
sehar.latif@ue.edu.pk

ABSTRACT

An entrepreneur is not developed in isolation, empirical studies followed this notion to explore the micro determinants of entrepreneurship. This study widens the scope and assesses the role of external factors like institutional quality of the country in determining the registration of new business. Different dimensions of institutional quality are expected to play their unique role in facilitating or restricting the process of becoming and entrepreneur. Based on the analysis across different development groups, the results showed that regulation compliance tends to discourage the entrepreneurship process.

1. INTRODUCTION

Entrepreneurial activities can be comprehensively analyzed through the lens of political economy. This frame of reference is substantial in gauging the importance of institutional quality. In the capitalist system, specific functions are particularly assigned to entrepreneurship productivity; nonetheless, the social aspects that add value to entrepreneurial activity should not be ignored (Minniti, 2008). There are three main dimensions of entrepreneurship that are of pertinence when a political-economy framework is employed. The dynamism of entrepreneurship stays intact when it is adaptable to the institutions' contextual underpinnings that act in an overarching manner. When the scenario is favorable, it can be mobilized as a vital pathway towards growth (Douhan & Henrekson, 2007). Better institutions are coined with increase the economic capacity of the indivdiuals (Hassan, Bukhari & Arshed, 2020).

In order to understand the role of institutional quality in accelerating entrepreneurial activities, there is a need to explore the existing evidence and barriers that obstruct the entrepreneurial activities. The focus of the current study is to explore the role of institutional quality in accelerating entrepreneurial activities. Moreover, the aim is to explore the relationship of institutional quality (legal) with entrepreneurial activities in term of both overall and in different cases of very high human development countries, high human development countries, medium human development countries, and low human development countries. This study will help the policymakers enhance the institutions regulatory and structural quality linked to the core institutional helix to develop the entrepreneurial activity.

2. LITERATURE REVIEW

2.1 Legal Institutional Quality

Voice and Accountability

Several theoretical frameworks regarding the functioning of the corporate sector and governance have been used to investigate the relationship between corporate functions and governmental entities (Nadeem et al., 2020; Silvestre & Ţîrcă, 2019). The functioning of proper legislative bodies and the presence of a valid legal system concerning corporate entities determine the economic growth of a nation (Mengistu & Adhikary, 2011). The need for accountability and governance are the two key contributing factors that determine the functioning capacity of the economic or corporate sector in a state (Wei, 2000). Many theorists have postulated different theoretical models to explain tax policies foreign funding FDI and corporate and legislative sphere interacting directly.

Legislative functionaries have deeply impacted the corporate entities' function along with state-legislative bodies in a country (Globerman & Shapiro, 2003). The presence of military or involvement of non-state actors greatly hinders the economic growth of any nation.

Rule of Law

Certain key elements determine the proper functioning of the corporate sector in any nation. Foreign funding and foreign direct investment can only be made if certain elements are present in the economy (Mengistu & Adhikary, 2011). The presence of certain points in the economic or corporate sector can increase its growth or hinder it. The availability of resources and peace along with law and legislative bodies functioning properly determine the economic growth of a nation. The presence of certain elements in the business sphere determines the GDP of any business: The presence of legislative functions and accountability in a government and proper governance determines business elements in terms of growth. State legislative bodies and corporate entities must develop a functioning relationship to determine a fair tax policy (Chakrabarti, 2001). Tax evasion is a major concern for various legal bodies and cooperative functioning entities in third world economies. A number of business theorists have proposed alternative business models to elaborate on corporate entities' functioning about tax policies in respective states (Gamso & Grosse, 2020).

3. DATA, METHODOLOGY AND RESULTS

3.1 Research Hypothesis

The current study aims to investigate the role of institutional quality in accelerating entrepreneurial activity in selected countries in the world and the sub-groups of very high human development countries, high human development countries, medium human development countries, and low human development countries. In order to do that the current study tests the following hypotheses.

H$_1$: There is a significant relationship between voice and accountability and entrepreneurial activity as measured by the new business density.

H$_2$: There is a significant relationship between rule of law and entrepreneurial activity measured by the new business density.

3.2 Theoretical Model

Figure 2 illustrates the theoretical framework of the study. Here different dimensions of institutional quality are assessed against the new business across the world.

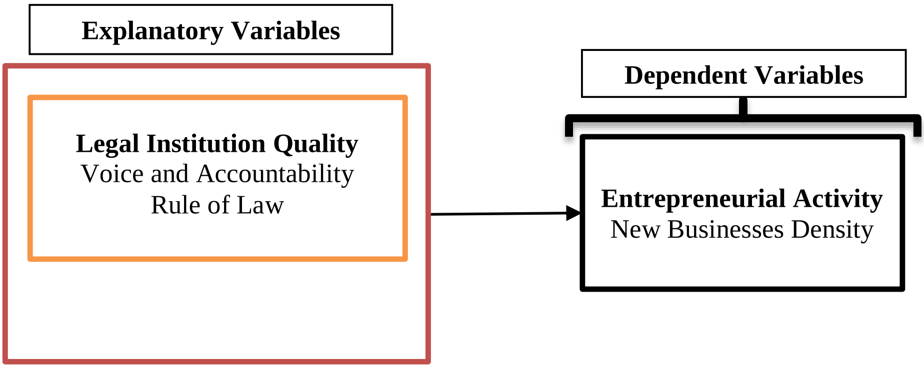

Figure 2: Theoretical Model

3.3 Research Design

The current study uses the deductive approach in order to test the hypothesis in case of both overall countries in the world and across different sub-groups based on their development level. The list of countries is taken from Arshed, Hanif, Aziz Croteau, (2021). That development level was provided by World Bank (2000). To do the analysis, secondary data has been used (Gabriel, 2013) in the regression analysis.

3.4 Data, Sample, and Variable Sources

The annual time series data for all explanatory variables and dependent variable (new business density) from years 2006 to 2016 was collected from World Development Indicators (2020). Table 1 shows the variable names, codes, and their sources.

Table 1
Variable Names, Codes, Definitions, and Sources

Variable Names	Codes	Explanatory / Dependent Variables	Sources
New Business Density	nbr	Dependent Variable: Entrepreneurial Activity	World Development Indicator
Rule of Law	rl	Explanatory Variable: Institutional quality	Worldwide Governance Indicator
Voice and Accountability	va	Explanatory Variable: Institutional quality	Worldwide Governance Indicator

3.5 Estimation Equation and Approach

Since the current paper is using a panel time series data where each variable changes across time and country, so Panel Generalized Least Square was chosen to investigate the role of institutional quality in accelerating entrepreneurial activity in case of both overall countries and in sub-groups: very high human development countries, high human development countries, medium human development countries, and low human development countries (Miller & Startz, 2018).

3.6 Results

3.6.1 Estimation Results

The focus of this section will be on the estimation results of New Business Density in the case of all those countries for which the data are available. Thus, only 112 countries could be used in the analysis, and the rest were not used due to the lack of data availability from some countries.

Table 2
Panel FLGS Estimates (Dependent Variable, nbr).

Variables	Overall	Very High HDI	High HDI	Medium HDI	Low HDI
	Coef. (Prob.)	Coef. (Prob.)	Coef. (Prob.)	Coef. (Prob.)	Coef. (Prob.)
RL	-0.865 (0.00)	-4.541 (0.00)	-0.134 (0.60)	-0.598 (0.00)	0.233 (0.12)
VA	-0.212 (0.00)	0.401 (0.01)	0.127 (0.26)	0.392 (0.00)	0.089 (0.20)
Cons	2.112 (0.00)	2.19 (0.00)	2.167 (0.00)	0.674 (0.00)	0.856 (0.00)
Obs	1075	532	306	144	93
Wald	2259 (0.00)	300 (0.00)	229 (0.00)	306 (0.00)	112 (0.00)
Key: New Business Density= nbr, Rule of Law= rl, and Voice and Accountability= va					

The results show that a one percent increase in the rule of law (rl) is associated with -0.864 percent decrease in New Business Density (nbr). The indirect relationship is that when the regulatory quality is too tough and rigorous, it is difficult for people to fulfill the requirement and go through the whole process before performing any business activity. So, it will hinder the development of entrepreneurial activity in a country where the rule of law is strictly implemented.

The results show that a one percent increase in voice and accountability (va) is associated with -0.211 percent decrease in New Business Density (nbr), where the probability value ($p<0.05$) confirmed that there is a negative and significant effect in case of all countries.

4. COMMENTS AND CONCLUSION

Overall, this study demonstrated that there is an indirect and significant relationship between legal institution quality and entrepreneurial activity in case of all countries in the world. Moreover, there is a direct and significant relationship between political institutional quality and entrepreneurial activity in all countries in the world. And in the case of all countries in the world, there is an indirect and significant relationship between governance effectiveness and entrepreneurial activity for the role of economic, institutional quality. There is a direct and significant relationship between regulatory quality and entrepreneurial activity.

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VIBRATION ANALYSIS OF FOUR LAYERED CYLINDRICAL SHELL FOR CANTILEVER EDGE AND WITH RING SUPPORT

Muhammad Omair Rasool, Madiha Ghamkhar[§]**,**
Saima Noreen and **Bushra Sadia**
Department of Mathematics and Statistics, University of Agriculture,
Faisalabad, Pakistan
[§]Email: madiha.ghamkhar@uaf.edu.pk

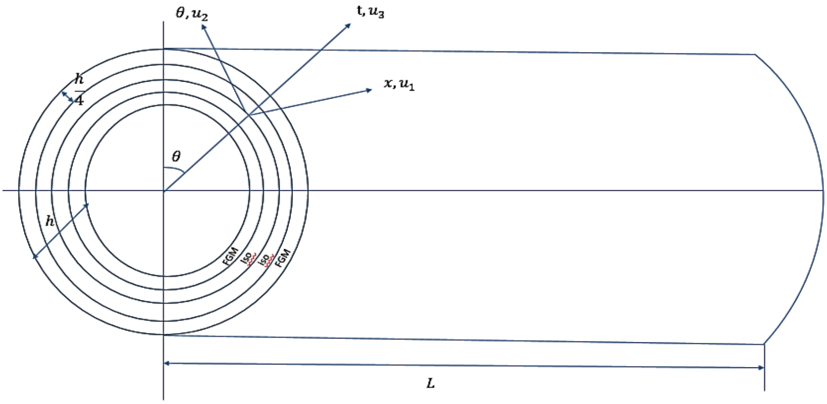
ABSTRACT

In this work vibration behavior and natural frequencies of a layered cylinder are observed and the layers are of functionally graded and isotropic materials. Vibrations of four-layered cylindrical shells by a ring assist over their length is investigated in this research. The outermost two layeres are made of isotropic material, whilst the internal two layeres are made of FGM. Stainless steel and nickel are used in the interior FGM layers. Aluminum and zirconia are used in the outer two isotropic layers. The Rayleigh-Ritz technique is used to collect the shell frequency equation in Sander's shell theory. The functionally graded material composition of the centre layer is sorted using three volume fraction laws (VFL), which are represented by mathematical formulations of polynomial, exponential and trigonometric functions. For the regularly informed inquiry, natural frequencies (NFs) are obtained under two boundary conditions: simply supported-simply supported and clamped-clamped.

INTRODUCTION

In structural analysis, a cylindrical shell is an important feature. Aside from their possible application, many mechanical aspects of these shell types are investigated, one of which is shell vibration. This study of such shells is essential in fields of technology such as liquid gases under high pressure, nuclear power plants, pipeline networks, and other maritime and aircraft systems. Zhi et al. [1] compare the frequency parameters with accurate three-dimensional linear elasticity analysis. The behaviour of natural frequencies was also studied in the vibrational analysis of FGM. Based on the Goldenviezer-Novozhilov shell principle, Iqbal et al. [2] investigated new precise programmes for the motions of the circular cylindrical shell with intermediate ring support and discussed the vibrational analysis of cylindrical shell and natural frequencies under various boundary conditions. We will look at power law, sigmoid, and exponential FGMs, as well as three-dimensional anisotropic elasticity basic equations, from which a time visibility with differential equations is formed in a coherent matrix form, as recommended by Iqbal et al. [3]. The effects of P-FGM versus S-FGM versus E-FGM are compared. Li et al. [4] proposed four sets of in-plane boundary conditions for the strictly assisted FG cylindrical shell for free vibration analysis. Physical properties that were previously thought to be temperature-dependent and 3 Theoretical considerations: A coordinate mechanism of cylinders (x, θ, t)

is represented as the shell the centre reference surface with x, θ and t is the coordinates of axial, angular and thickness correspondingly.



longitudinal

Figure 1: A Ring Support Runs the Length of the Structure in CS Geometry

The strain energy relation is : $\mathbb{k} = \frac{1}{2}\int_0^{L}\int_0^{2\pi} \{D\}^{'} [\mathbb{Z}]\{D\} R d\theta dx$

where $\{D\}^{'} = \{\tau_1, \tau_2, \Upsilon, \mathrm{I}_1, \mathrm{I}_2, 2\delta\}$ and the matrix [C] are represents as follows,

$$[C] = \begin{bmatrix} \ell_{11} & \ell_{12} & 0 & \exists_{11} & \exists_{12} & 0 \\ \ell_{12} & \ell_{22} & 0 & \exists_{12} & \exists_{22} & 0 \\ 0 & 0 & \ell_{66} & 0 & 0 & \exists_{66} \\ \exists_{11} & \exists_{12} & 0 & \ni_{11} & \ni_{12} & 0 \\ \exists_{12} & \exists_{22} & 0 & \ni_{12} & \ni_{22} & 0 \\ 0 & 0 & \exists_{66} & 0 & 0 & \ni_{66} \end{bmatrix}, \qquad (i, j = 1, 2, 6),$$

where τ_1, τ_2, γ expressed the strains that consists of reference surface and $\{\mathrm{I}_1, \mathrm{I}_2, \delta\}$ shows curvatures and ℓ_{ij} $\exists_{ij}$, and $\ni_{ij}$, shows the extensional coupling & bending stiffness (Loy et al. 1999).They are defined as:

$$\{\ell_{ij}, \exists_{ij}, \ni_{ij}\} = \int_{-\frac{H}{2}}^{\frac{H}{2}} \Theta_{ij} \left\{1, z, z^2\right\} dz \tag{1}$$

Isotropic materials reduced stiffness Θ_{ij} is stated as,

$$\Theta_{11} = \frac{E}{(1-\lambda^2)} = \Theta_{22}\,,\ \Theta_{12} = \frac{\lambda E}{1-\lambda^2} \text{ and } \Theta_{66} = \frac{E}{2\left(1-\lambda^2\right)} \tag{2}$$

The matrix $\exists_{ij} = 0$ for isotropic circular shaped CS and $\exists_{ij} \neq 0$ for a FG cylindrical shell. Solving above expression then result (J) is written as

$$\mathrm{J} = \frac{1}{2}\int_0^L\int_0^{2\pi}\Big\{\ell_{11}\tau_1^2 + \ell_{22}\tau_2^2 + 2\ell_{11}\tau_1\tau_2 + \ell_{66}\gamma^2 + 2\exists_{11}\tau_1\mathrm{I}_1$$
$$+2\exists_{12}\tau_1\mathrm{I}_2 + 2\exists_{12}\tau_2\mathrm{I}_1 + 2\exists_{22}\tau_2\mathrm{I}_2 + 4\exists_{66}\gamma\delta + \ni_{11}\mathrm{I}_1^2 \tag{3}$$
$$+\ni_{22}\mathrm{I}_2^2 + 2\ni_{12}\mathrm{I}_1\mathrm{I}_2 + 4\ni_{66}\gamma^2\Big\}Rd\theta dx$$

where;

$$\{\tau_1, \tau_2, \gamma\} = \left\{\frac{\partial u_1}{\partial x}, \frac{1}{R}\left(\frac{\partial u_2}{\partial \theta} + u_3\right), \left(\frac{\partial u_2}{\partial x} + \frac{1}{R}\frac{\partial u_1}{\partial \theta}\right)\right\} \tag{4}$$

$$\{\mathrm{I}_1, \mathrm{I}_2, \delta\} = \left\{-\frac{\partial^2 u_3}{\partial x^2}, -\frac{1}{R^2}\left(\frac{\partial^2 u_3}{\partial \theta^2} - \frac{\partial u_2}{\partial \theta}\right), -\frac{2}{R}\left(\frac{\partial^2 u_3}{\partial x \partial \theta} - \frac{3}{4}\frac{\partial u_2}{\partial x} + \frac{1}{4R}\frac{\partial u_1}{\partial \theta}\right)\right\}, \tag{5}$$

Putting value from (4) and (5) into equation (3).Shell kinetic energy is stated as

$$\lambda = \frac{1}{2}\int_0^L\int_0^{2\pi}\rho_t\left[\left(\frac{\partial u_1}{\partial t}\right)^2 + \left(\frac{\partial u_2}{\partial t}\right)^2 + \left(\frac{\partial u_3}{\partial t}\right)^2\right]Rd\theta dx \tag{6}$$

where

$$\rho_t = \int_{-\frac{H}{2}}^{\frac{H}{2}} \rho\, dz, \tag{7}$$

where ρ is the density of mass. The kinetic and strain energies are used to define the Lagrange energy functional (η) for a CS, which is:

$$\eta = \lambda - \Theta \tag{8}$$

Numerical Procedure

The Rayleigh-Ritz parameter controls the normal frequencies of CS. The displacement domains are assumed as

$$u_1(x,\theta,t) = x_n U_1(x)\cos(m\theta)\,sin(\varpi t),$$
$$u_2(x,\theta,t) = y_n U_2(x)\sin(m\theta)\,cos(\varpi t)$$
$$u_3(x,\theta,t) = z_n U_3(x)\cos(m\theta)\,sin(\varpi t).$$

where x_n, y_n and z_n show the amplitudes of vibration in the x, θ and t direction, the axial (m) and circumferential wave numbers (n) of mode shapes. (ω) signifies the angular vibration frequency of the shell wave, while $U(x), V(x)$ and $W(x)$, Is the longitudinal, circumferential and transverse directions, signifies the axial model dependency, respectively.

$\phi(x)$ is the axial function that meets the geometric edge constraint and is defined as

$$\begin{aligned}\varphi(x) &= \beta_1 \cosh(\mu_m x) + \beta_2 \cos(\mu_m x) \\ &\quad -\sigma_m \beta_3 \sinh(\mu_m x) + \sigma_m \beta_4 \sin(\mu_m x)\end{aligned} \tag{9}$$

μ_m Roots of some transcendental equations, σ_m Parameters which depends on the values of μ_m and β_i $(i = 1,2,3,4)$ are change with respect to the edge conditions.

Non-dimensional parameters are used for generalization as;

$$\mathrm{U}_1 = \frac{D(x)}{h}, U_2 = \frac{E(x)}{h}, U_3 = \frac{F(x)}{R} \tag{10}$$

$$\mathrm{A}_{ij} = \frac{p_{ij}}{h}, \ \mathrm{B}_{ij} = \frac{q_{ij}}{h^2}, \ \mathrm{C}_{ij} = \frac{r_{ij}}{h^3},$$

$$\psi = R/L \ , \zeta = h/R, X = x/L \ , \rho_t = p_t / h$$

Now expression becomes:

$$u_1(x,\theta,t) = h x_n U_1 \cos(m\theta) \sin(\varpi t),$$
$$u_2(x,\theta,t) = h y_n U_2 \sin(m\theta) \cos(\varpi t)$$
$$u_3(x,\theta,t) = h z_n U_3 \cos(m\theta) \sin(\varpi t).$$

Formation of eigen value frequency equation

Lagrangian energy functional $\eta_{lagrange}^{\max}$ is minimized with respect to wave amplitudes x_n, y_n and z_n as fellow,

$$\frac{\partial \eta_{lagrange}^{\max}}{\partial x_n} = \frac{\partial \eta_{lagrange}^{\max}}{\partial y_n} = \frac{\partial \eta_{lagrange}^{\max}}{\partial z_n} = 0 \tag{11}$$

The resulted relation define in the form of matrix as

$$\left\{[G] - \Omega^2 [J]\right\} \hat{X} = 0 \tag{12}$$

where

$$\Omega^2 = R^2 \omega^2 \rho_t, \tag{13}$$

The rigidity [G] and mass matrices [J] of the cylindrical shell.

Classification of Materials

A cylindrical shell is regarded in this study to be made up of four layers, with the internal layers made of isotropic material and the exterior layers made of FG materials nickel and stainless steel. The following relationships indicate the volume fractions of the shell outer layers formed from two parts using trigonometric volume fraction law (VFL).

$$V_{f1}=\sin^2\left[\left\{\frac{z-h_1}{h_2-h_1}\right\}^N\right],\quad V_{f2}=\cos^2\left[\left\{\frac{z-h_1}{h_2-h_1}\right\}^N\right],\quad 0\le N\le\infty$$

$V_{f1}+V_{f2}=1$, so relation is satisfied where h shell thickness and N power law exponent and each layer thickness is $h/4$. Material parameters: E_1,λ_1,η_1 is for Nickel, E_1,λ_1,η_1 for Zirconia, E_3,λ_3,η_3 for stainless steel, E_4,λ_4,ρ_4 is for Aluminum, E_5,λ_5,η_5 for stainless steel and E_6,λ_6,η_6 for Zirconia. Then the effective material quantities:

$$E_{fgm1}=[E_1-E_2]\sin^2\left[\left\{\frac{z-h_1}{h_2-h_1}\right\}^N\right]+E_2,\ \lambda_{fgm1}=[\lambda_1-\lambda_2]\sin^2\left[\left\{\frac{z-h_1}{h_2-h_1}\right\}^N\right]+\lambda_2$$

$$\eta_{fgm1}=[\eta_1-\eta_2]\sin^2\left[\left\{\frac{z-h_1}{h_2-h_1}\right\}^N\right]+\eta_2,\ E_{fgm2}=[E_4-E_5]\sin^2\left[\left\{\frac{z-h_3}{h_4-h_3}\right\}^N\right]+E_5$$

$$\lambda_{fgm2}=[\lambda_4-\lambda_5]\sin^2\left[\left\{\frac{z-h_3}{h_4-h_3}\right\}^N\right]+\lambda_5,\ \eta_{fgm2}=[\eta_4-\eta_5]\sin^2\left[\left\{\frac{z-h_3}{h_4-h_3}\right\}^N\right]+\eta_5$$

For expression above at $z=-\frac{h}{2},\frac{h}{2}$

$$E_{fgm1}=E_2,\lambda_{fgm1}=\lambda_2,\eta_{fgm1}=\eta_2 \text{ and } E_{fgm2}=E_5,\lambda_{fgm2}=\lambda_5,\eta_{fgm2}=\eta_5$$

respectively.

The stiffness moduli are modified as:

$$\xi_{ij}=\xi_{ij}(FGM)+\xi_{ij}(iso)+\xi_{ij}(FGM)+\xi_{ij}(iso),$$
$$\psi_{ij}=\psi_{ij}(FGM)+\psi_{ij}(iso)+\psi_{ij}(iso)+\psi_{ij}(FGM),$$
$$\zeta_{ij}=\zeta_{ij}(FGM)+\zeta_{ij}(iso)+\zeta_{ij}(iso)+\zeta_{ij}(FGM)$$

where $i=1,2,6$.

RESULTS AND DISCUSSION

Four Layered Cylindrical Shell

To explain the accuracy and knowledge of the exact outcome, the current outcome is used by FGM and Isotropic CS for SS edge conditions. Rayleigh-Ritz method is used to achieve the current result. Three materials are used to make FG materials are known to be FGM cylindrical layers: Stainless steel, Zirconia and nickel. Two FGM variations for the form of substance are found.

Type 2: In type 2 FGM CS, inner surfaces are made with Zirconia and outer layers are composed of Nickel and Stainless Steel.

Configurations of shells:

Composition of four layered cylindrical shall made by FGM and isotropic type of material, define below in the table.

Table 1
Configurations of Shell Types

Types of Shell	First Layer FGM	Second Layer Isotropic	Third layer Isotropic	Fourth Layer FGM
Type 2	Zirconia, Nickel	Stainless Steel	Zirconia, Stainless Steel	Aluminum

Table 2
Comparison of Natural Frequency (Hz) against Circumferential Wave Number n for FGM Cylindrical Shell with s-s Boundary Condition when (m=2, L=20, h=0.02, R=1, a=0.1).

N	N=1	N=3	N=5	N=7	N=9	N=11
1	362.6197	359.9578	359.2090	359.2090	359.2090	359.2090
2	455.8980	452.9469	452.1372	452.1372	452.1372	452.1372
3	477.2085	474.1213	473.2755	473.2755	473.2755	473.2755
4	485.2101	482.0728	481.2149	481.2149	481.2149	481.2149
5	489.6049	486.4406	485.5777	485.5777	485.5777	485.5777
6	493.1596	489.9726	489.1063	489.1063	489.1063	489.1063
7	497.0994	493.8851	493.0148	493.0148	493.0148	493.0148
8	502.2050	498.9528	498.0760	498.0760	498.0760	498.0760
9	509.1086	505.8025	504.9154	504.9154	504.9154	504.9154
10	518.3821	515.0016	514.0989	514.0989	514.0989	514.0989

Table 3
Behavior of Natural Frequency (Hz) against Circumferential Wave Number n for FGM Cylindrical Shell with Simply Supported-Simply Supported Boundary Condition when (m=1, L=20, N=5, R=1, a=0.5)

N	h=0.001	h=0.005	h=0.007	h=0.01	h=0.05	h=0.07
1	4313.561	4314.657	4314.658	4314.659	4314.673	4314.682
2	779.3320	838.7948	838.7603	838.7167	839.0724	839.9035
3	787.4543	838.7012	838.6600	838.6395	843.0766	848.5493
4	789.8100	838.6410	838.6587	838.8150	855.6305	873.9499
5	790.8129	838.6853	838.8951	839.5266	883.1276	927.3678
6	791.3239	838.9250	839.5478	841.1367	932.4724	1019.1568
7	791.6158	839.4714	840.8337	844.0830	1009.7367	1156.0430
8	791.8012	840.4554	843.0066	848.8724	1009.7367	1156.0430
9	791.9362	842.0268	846.3550	856.0697	1262.0934	1571.3251
10	792.0536	844.3534	851.1977	866.2819	1438.9259	1846.7592

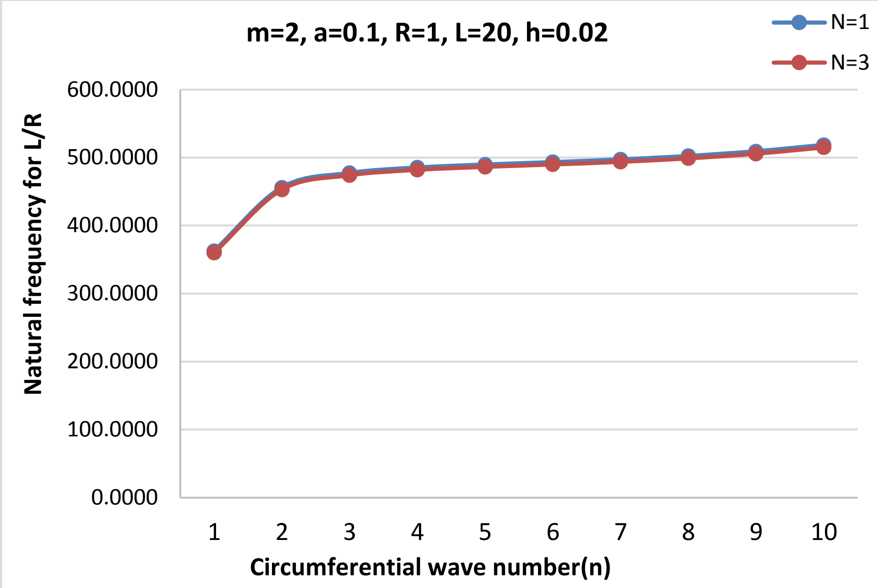


Figure 1: Variation in NF against n of Type-2 SS-SS Four Layer CS (m=2, a=0.1, h=0.002, R=1, L=20)

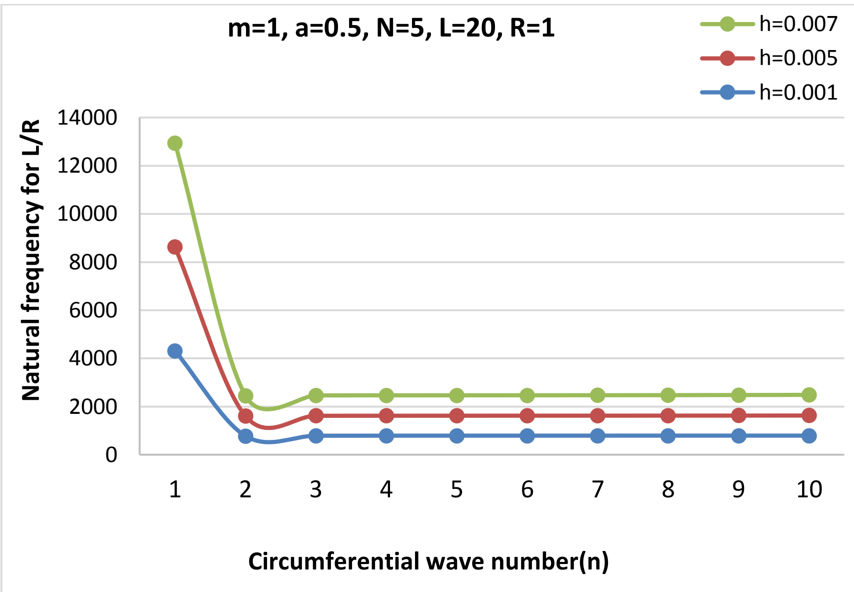


Figure 2: Variation in NF against n of Type-2 SS-SS Four Layer CS (m=1, a=0.5, N=5, R=1, L=20)

CONCLUSIONS

The purpose of this research was to examine the vibration properties of thin circular cylindrical shells made from functionally graded materials. FG cylindrical shell constituents constructed of Zirconia, Nickel, stainless Steel, and Zirconia, Stainless Steel, Aluminum have been taken. These four varieties are obtained by altering the configuration of these FG material layers. These elements in type 2 cylindrical shells fluctuate constantly, smoothly, and progressively along the thickness of the shell from the inner to the outside of the FG layer. The main goal was to use these ideas and techniques to obtain shell frequency equations. Based on Kirchhoff's assumptions, Love's thin shell theory is used to solve shell frequency equations. By employing the Lagrangaian energy relation, the Rayleigh Ritz approach is employed to tackle shell vibration problems. Axial modal dependence approximated by trigonometric functions for simply supported end conditions and characteristics beam functions for number clamped boundary conditions.

There was a direct increase in the value of N, L, m, a, h, and L/R to ensure that the minimum frequency occurs in wave numbers for all boundary circumstances. At varied conditions, the NFs dropped as the N increased. All of the outcomes are extremely similar. This technique could be used to investigate a variety of shell-related issues.

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VIBRATION FREQUENCY ANALYSIS OF FOUR LAYERED CYLINDRICAL SHELL FOR DIFFERENT EDGE CONDITIONS

Fatima Irshad, Madiha Ghamkhar§, Muhammad Omair Rasool
and **Abid Maqbool**
Department of Mathematics and Statistics
University of Agriculture, Faisalabad, Pakistan
§Email: madiha.ghamkhar@uaf.edu.pk

ABSTRACT

In this research, a four-layered cylindrical shell's vibration frequency analysis is studied, where the 1st and 4th layers are made of isotropic material and the 2nd and 3rd layers are of functionally graded materials. Sander's shell principle is used for the displacement relations of strain and curvature. Using the Rayleigh Ritz mathematical approach, the shell frequency equation is obtained. The characteristic beam functions are used to calculate axial model dependency. Under different edge conditions, results for natural frequencies against the circumferential wave number with increasing thickness, length and power exponent law are obtained for a four layered cylindrical shell with the help of the MATLAB software. Furthermore, the methodology accuracy is checked with various results.

1. INTRODUCTION

In the technology and engineering fields, cylindrical shells are important components. For its basic geometrical design, the vibrations of cylindrical shells are researched thoroughly. There are numerous studied on them in the open literature. Rayleigh (1882) examined Sophie's work on circular cylindrical shell vibration. Love (1888) presented the 1st linear shell theory. Further shell's theories are constructed from this idea by changing some physical concepts. Arnold and Warburton (1949) first investigated the vibration of cylindrical shells. Sharma and Johns (1971) analyzed the cylindrical shell oscillations under clamped-free boundary constraints with the Ritz method. Numerous approaches to the vibrational behavior of thin shells are being investigated by several researchers. These methods vary from the Rayleigh-Ritz-based energy methods to analytic approaches in which closed-form results are respectively employed. Scedel (1980) developed a different frequency formulation for a clamped-clamped, clamped-free and free-free cylindrical shell. Chung (1981) used Sander's shell equation to investigate cylindrical shell oscillations by clamping, freeing and simply supporting them. Lam and Loy (1995) used the Ritz technique to investigate the impact of constraints on the multi-layered cylindrical shell's vibration properties. Loy and Lam (1997) studied the thin cylindrical container's vibration using ring supports during the length of the container, which required no side rebound. In order to investigate radial buckling of cylindrical shells, Bhutta *et al.* (2015) studied the tri-laying cylindrical shell concept with an interlayer of FGM and extremely isotropic type layers. Li and Batra (2006) investigated the three-layer circular cylindrical shell's bending that is simply supported by an axially compressive force. Sofiyev *et al.* (2006) examined the oscillation study for the tri-layered conic shell, with an FGM central layer. Iqbal *et al.* (2009) used the method of wave propagation to evaluate the oscillations in the circular shells of functionally graded materials.

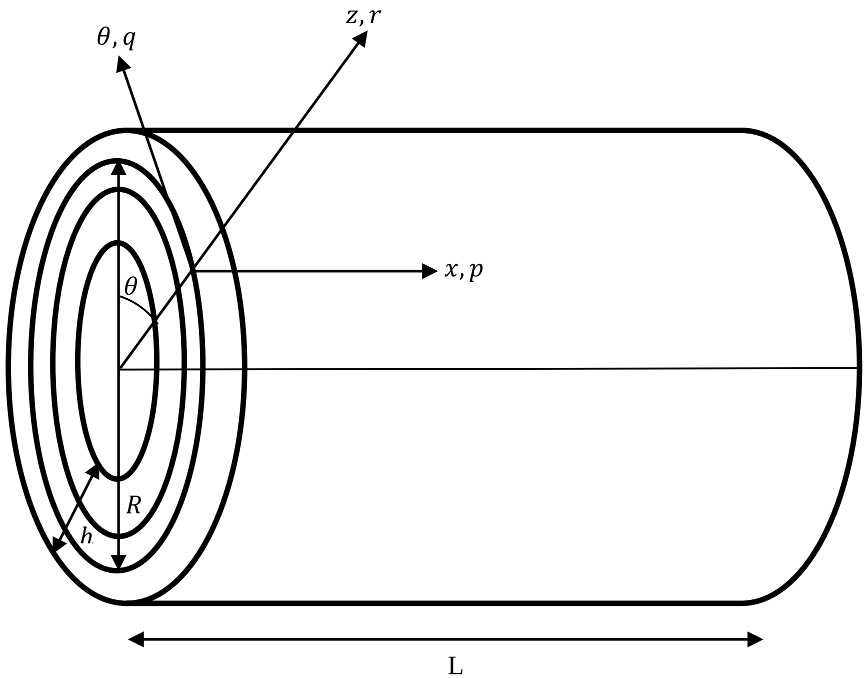

Figure 1: Four-Layered Cylindrical Shell Structure

2. THEORETICAL CONSIDERATIONS

Consider a four layered CS using radius R, thickness h and length L as shown in Figure 1. The central surface of the cylinder shape shell is oriented to an orthogonal co-ordinate scheme $(x, \theta \text{ and } z)$. Here x, θ and z lie in the axial, circumferential also radial shell ways. And shell displacements in x, θ and z directions are (p, q, r) respectively. The fundamental relationship between stress and strain is defined through generalized Hooke's law as follows:

$$[\sigma] = [Q][\varepsilon]. \tag{1}$$

Here the stress vector, reduced stiffness and strain vector are denoted by $[\sigma]$, $[Q]$ and $[\varepsilon]$. The vectors of stress plus strain are shown as:

$$[\sigma]=\begin{bmatrix}\sigma_x\\ \sigma_\theta\\ \sigma_{x\theta}\end{bmatrix},\qquad [\varepsilon]=\begin{bmatrix}\varepsilon_x\\ \varepsilon_\theta\\ \varepsilon_{x\theta}\end{bmatrix}. \tag{2}$$

Here, x, θ-direction stress σ_x, σ_θ and $x\theta$-direction shear stress $\sigma_{x\theta}$ are shown. In the same way x, θ-direction strain and $x\theta$-direction shear strain are denoted by ε_x, ε_θ and $\varepsilon_{x\theta}$. The reduced stiffness matrix has the following formula:

$$[Q]=\begin{bmatrix}Q_{11} & Q_{12} & 0\\ Q_{21} & Q_{22} & 0\\ 0 & 0 & Q_{66}\end{bmatrix}. \tag{3}$$

For ISO materials, reduced stiffness entries are stated as

$$Q_{11}=\frac{E}{1-\mu^2}, Q_{22}=\frac{E}{1-\mu^2}, Q_{12}=\frac{\mu E}{1-\mu^2}, Q_{66}=\frac{E}{2\left(1+\mu^2\right)}, \tag{4}$$

(Loy *et al.*, 1999).

Here, Young's modulus then Poisson ratio are represented by E and μ. By using shell theory of Love, the strain and curvature relations are defined as:

$$\varepsilon_x=\varepsilon_1+z\tau_1,\ \ \varepsilon_\theta=\varepsilon_2+z\tau_2,\ \ \varepsilon_{x\theta}=\varepsilon_{12}+2z\tau_{12}. \tag{5}$$

where, $\varepsilon_1, \varepsilon_2, \varepsilon_{12}$ are the surface relations of the strains and $\tau_1, \tau_2, \tau_{12}$ are the surface relations of the curvatures. By using the expression (5) in expressions (2), (3) and (1). Then we have

$$\begin{aligned}\sigma_x&=Q_{11}\left(\varepsilon_1+z\tau_1\right)+Q_{12}\left(\varepsilon_2+z\tau_2\right),\\ \sigma_\theta&=Q_{21}\left(\varepsilon_1+z\tau_1\right)+Q_{22}\left(\varepsilon_2+z\tau_2\right),\\ \sigma_{x\theta}&=Q_{66}\left(\varepsilon_{12}+2z\tau_{12}\right).\end{aligned} \tag{6}$$

A cylindrical shell's forces then moment resultants are described as:

$$\begin{aligned}\{N_x, N_\theta, N_{x\theta}\}&=\int_{-\frac{h}{2}}^{\frac{h}{2}}\{\sigma_x,\sigma_\theta,\sigma_{x\theta}\}dz,\\ \{M_x, M_\theta, M_{x\theta}\}&=\int_{-\frac{h}{2}}^{\frac{h}{2}}\{\sigma_x,\sigma_\theta,\sigma_{x\theta}\}zdz.\end{aligned} \tag{7}$$

Here force components in axial, circumferential and radial direction are indicated as N_x, N_θ and $N_{x\theta}$. Also moment components in axial, circumferential and radial direction are presented such as M_x, M_θ and $M_{x\theta}$. By applying expressions (7) and (6), then

$$[N]=[O][\tau]. \tag{8}$$

where, $[N]$, $[\tau]$ and $[O]$ are defined as

$$[N]=\begin{bmatrix} N_x \\ N_\theta \\ N_{x\theta} \\ M_x \\ M_\theta \\ M_{x\theta} \end{bmatrix}, [\tau]=\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_{12} \\ \tau_1 \\ \tau_2 \\ 2\tau_{12} \end{bmatrix}, [O]=\begin{bmatrix} A_{11} & A_{12} & 0 & B_{11} & B_{12} & 0 \\ A_{12} & A_{22} & 0 & B_{12} & B_{22} & 0 \\ 0 & 0 & A_{66} & 0 & 0 & B_{66} \\ B_{11} & B_{12} & 0 & D_{11} & D_{12} & 0 \\ B_{12} & B_{22} & 0 & D_{12} & D_{22} & 0 \\ 0 & 0 & B_{66} & 0 & 0 & D_{66} \end{bmatrix}. \tag{9}$$

Here the extensional stiffness A_{ij}, coupling stiffness B_{ij} as well as bending stiffness D_{ij} is expressed respectively, $i=1,2,6;$

$$A_{ij} = \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{ij} dz \ , B_{ij} = \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{ij} z dz \ , D_{ij} = \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{ij} z^2 dz \ . \tag{10}$$

2.1 Construction of Strain Energy

The strain energy υ formulation for CS is defined as

$$\upsilon=\frac{1}{2}R\int_0^L\int_0^{2\pi}[\tau]^{'}[O][\tau]d\theta dx. \tag{11}$$

By using the expression (9), υ is reviewed like

$$\begin{aligned}\upsilon=\frac{R}{2}\int_0^L\int_0^{2\pi}\Big\{&A_{11}\varepsilon_1^2+A_{22}\varepsilon_2^2+2A_{11}\varepsilon_1\varepsilon_2+A_{66}\varepsilon_{12}^2+2B_{11}\varepsilon_1\tau_1\\ &+2B_{12}\varepsilon_1\tau_2+2B_{12}\varepsilon_2\tau_1+2B_{22}\varepsilon_2\tau_2+4B_{66}\varepsilon_{12}\tau+D_{11}\tau_1^2\\ &+D_{22}\tau_2^2+2D_{12}\tau_1\tau_2+4D_{66}\tau_{12}^2\Big\}d\theta dx.\end{aligned} \tag{12}$$

Shell kinetic energy κ_s is stated as:

$$\kappa_s=\frac{R}{2}\int_0^L\int_0^{2\pi}\rho_t\left[\left(\dot{p}\right)^2+\left(\dot{q}\right)^2+\left(\dot{r}\right)^2\right]d\theta dx. \tag{13}$$

where the dot is just above the quantity, the time derivative of the amount is indicated. Mass density shell ρ_s and mass density for any duration of unit ρ_t is illuminated below and where ρ_s is the mass density shell.

$$\rho_t = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho_s dz. \tag{14}$$

2.2 Displacement Relations of Strain and Curvature

Various shell theories were introduced to determine the cylindrical shell's free vibration. Love's shell theory is basic first-order. Budiansky and Sanders (1963) introduced the displacement relations of strain and curvature from the shell theory of first-order. According to this theory, displacement relations of strain and curvature are given by

$$\begin{aligned}\{\varepsilon_1,\varepsilon_2,\varepsilon_{12}\} &= \left\{\frac{\partial p}{\partial x},\frac{1}{R}\left(\frac{\partial q}{\partial\theta}+r\right),\left(\frac{\partial q}{\partial x}+\frac{1}{R}\frac{\partial p}{\partial\theta}\right)\right\},\\ \{\tau_1,\tau_2,2\tau_{12}\} &= \left\{-\frac{\partial^2 r}{\partial x^2},-\frac{1}{R^2}\left(\frac{\partial^2 r}{\partial\theta^2}-\frac{\partial q}{\partial\theta}\right),-\frac{2}{R}\left(\frac{\partial^2 r}{\partial x\partial\theta}-\frac{3}{4}\frac{\partial q}{\partial x}+\frac{1}{4R}\frac{\partial p}{\partial\theta}\right)\right\}.\end{aligned} \tag{15}$$

2.3 Lagrange Energy Functional

The Lagrange energy functional L_f for CS is shown as:

$$L_f = \kappa_s - \upsilon. \tag{16}$$

2.4 Axial Modal Dependency

Nonlinear displacement deform formulas are viewed as just a space and time factors mix. That leads to ordinary differential equations set for the axial space variable's three unknown functions. The axial modal dependency is move toward by numerous forms of functions. Now relationships become supposed in the displacement fields:

$$\begin{aligned} p(x,\theta,t) &= X_m P(x)\cos(n\theta)\sin\omega t,\\ q(x,\theta,t) &= Y_m Q(x)\sin(n\theta)\cos\omega t,\\ r(x,\theta,t) &= Z_m R(x)\cos(n\theta)\sin\omega t.\end{aligned} \tag{17}$$

Here X_m, Y_m and Z_m represent the vibration amplitudes in x, θ and z direction respectively. The mode form axial and circumferential wave numbers are indicated by m and n. The shell vibration angular frequency is denoted by ω. $P(x)$, $Q(x)$ and $R(X)$ are indicates the axial model dependency in the longitudinal, circumferential also transverse directions. Here we take

$$P(x) = \frac{d}{dx}(\zeta(x)),\ Q(x) = \zeta(x),\ \ R(x) = \zeta(x). \tag{18}$$

The beam function $\zeta(\mathrm{x})$ is taken as in the following form

$$\zeta(\mathrm{x}) = \beta_1 \cosh(\eta_m x) + \beta_2 \cos(\eta_m x) - \varsigma_m \beta_3 \sinh(\eta_m x) - \varsigma_m \beta_4 \sin(\eta_m x). \tag{19}$$

Here values of β_i are modified the conditions of the edge $(i = 1, 2, 3, 4)$, η_m indicate origins of such transcendental equations and ς_m parameters based on η_m values.

2.5 Minimized Lagrangian Energy Functional

The reduced Lagrangian energy functional with concerning the amplitudes X_m, Y_m and Z_m of vibration for example,

$$\frac{\partial L_{f(\max)}}{\partial X_m} = \frac{\partial L_{f(\max)}}{\partial Y_m} = \frac{\partial L_{f(\max)}}{\partial Z_m} = 0. \tag{20}$$

The form of the matrix of shell frequency equations for cylindrical shell is expressed as:

$$\left\{ \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{12} & c_{22} & c_{23} \\ c_{13} & c_{23} & c_{33} \end{bmatrix} - h_s \rho \omega^2 \begin{bmatrix} I_2 & 0 & 0 \\ 0 & I_4 & 0 \\ 0 & 0 & I_4 \end{bmatrix} \right\} \begin{bmatrix} X_m \\ Y_m \\ Z_m \end{bmatrix} = 0. \tag{21}$$

2.6 Classifications of Materials

In four-layered cylindrical shell, the interior and exterior layers are made-up from isotropic materials while the central layers are fabricated from FG materials: nickel, zirconia, stainless-steel and aluminum. The shell thickness is split into four layers. Internal, intermediates and exterior layers thickness are h_1, h_2, h_3 and h_4 respectively. To each layer is of thickness $\frac{h}{4}$. To observe the effect of material parameters for FGMs, the volume fraction law is applied to cylindrical shells. The trigonometry volume fraction law of two components of a shell that has first FGM layer for a cylindrical shell is defined by the relation:

$$V_{f1} = \sin^2\left[\left(\frac{z - h_1}{h_2 - h_1}\right)^N\right], \quad V_{f2} = \cos^2\left[\left(\frac{z - h_1}{h_2 - h_1}\right)^N\right], \quad 0 \le N \le \infty. \tag{22}$$

These relationships fulfill the VFL i.e. $V_{f2} + V_{f1} = 1$. Then the effective resultant material quantities $\mathrm{E}^{(FGM)}, \mu^{(FGM)}$ and $\rho_s^{(FGM)}$ are expressed as

$$E^{(FGM)2} = \left(\mathrm{E}_2 - \mathrm{E}_3\right)\sin^2\left(\frac{z+h/4}{h/4}\right)^N + \mathrm{E}_3, E^{(FGM)3} = \left(\mathrm{E}_4 - \mathrm{E}_5\right)\sin^2\left(\frac{z}{h/4}\right)^N + \mathrm{E}_5,$$
$$\mu^{(FGM)2} = \left(\mu_2 - \mu_3\right)\sin^2\left(\frac{z+h/4}{h/4}\right)^N + \mu_3, \mu^{(FGM)3} = \left(\mu_4 - \mu_5\right)\sin^2\left(\frac{z}{h/4}\right)^N + \mu_5,$$
$$\rho_s^{(FGM)2} = \left(\rho_{s2} - \rho_{s3}\right)\sin^2\left(\frac{z+h/4}{h/4}\right)^N + \rho_{s3}, \rho_s^{(FGM)3} = \left(\rho_{s4} - \rho_{s5}\right)\sin^2\left(\frac{z}{h/4}\right)^N + \rho_{s5}. \tag{23}$$

For the 2nd FGM layer, the above expressions at $z = -h/4$ and $z = 0$ becomes:

$$\mathrm{E}^{(FGM)} = \mathrm{E}_3,\ \mu^{(FGM)} = \mu_3,\ \rho_s^{(FGM)} = \rho_{s3}. \tag{24}$$

and

$$\begin{aligned} E^{(FGM)2} &= \left(\mathrm{E}_2 - \mathrm{E}_3\right)\sin^2(1) + \mathrm{E}_3, \\ \mu^{(FGM)2} &= \left(\mu_2 - \mu_3\right)\sin^2(1) + \mu_3, \\ \rho_s^{(FGM)2} &= \left(\rho_{s2} - \rho_{s3}\right)\sin^2(1) + \rho_{s3}. \end{aligned} \tag{25}$$

For the 3rd FGM layer, the above expressions at $z = 0$ and $z = h/4$ becomes:

$$\mathrm{E}^{(FGM)} = \mathrm{E}_5,\ \mu^{(FGM)} = \mu_5,\ \rho_s^{(FGM)} = \rho_{s5}. \tag{26}$$

and

$$\begin{aligned} E^{(FGM)3} &= \left(\mathrm{E}_4 - \mathrm{E}_5\right)\sin^2(1) + \mathrm{E}_5, \\ \mu^{(FGM)3} &= \left(\mu_4 - \mu_5\right)\sin^2(1) + \mu_5, \\ \rho_s^{(FGM)3} &= \left(\rho_{s4} - \rho_{s5}\right)\sin^2(1) + \rho_{s5}. \end{aligned} \tag{27}$$

As a result, the shell is completely made from stainless steel at $z = -h/4$ for the 2nd layer and nickel at $z = 0$ for the 3rd layer, and the material properties are combination of zirconia and stainless steel for the 2nd layer at $z = 0$ and the material properties are combination of aluminum and nickel for the 3rd layer at $z = h/4$. The modified stiffness moduli illustrated as:

$$\begin{aligned} A_{ij} &= A_{ij}(iso) + A_{ij}(FGM) + A_{ij}(FGM) + A_{ij}(iso), \\ B_{ij} &= B_{ij}(iso) + B_{ij}(FGM) + B_{ij}(FGM) + B_{ij}(iso), \\ D_{ij} &= D_{ij}(iso) + D_{ij}(FGM) + D_{ij}(FGM) + D_{ij}(iso). \end{aligned} \tag{28}$$

Here $i, j = 1,2,6$ and (iso) denotes the 1st and 4th isotropic layers then (FGM) denotes the 2nd and 3rd functionally graded material layers.

3. RESULTS AND DISCUSSION

The variation in natural frequencies (NFs) results are designed for different thickness, length and power exponent law against the circumferential wave number (n) for a clamped-clamped (C-C) and simply supported-simply supported (SS-SS) four layered cylindrical shell (CS) that illustrated in tables and figures. In table 1, given the variation in NFs for the C-C four layered CS when $(m=2,\ N=10,\ L=20,\ R=1)$ is applied. For n=1, 2 and 3, the natural frequency increases horizontally from 40.2391 to 40.2390, 13.8976 to 15.0363 and 6.6687 to 17.6559 by increasing the thickness as 0.001, 0.005, 0.007, 0.01 and 0.05. When circumferential wave numbers increase vertically from 1 to 5 then natural frequency increases horizontally from 40.2391 to 74.3574 by increasing the thickness as 0.001, 0.005, 0.007, 0.01 and 0.05. In table 2, given the variation in NFs for C-C four layered CS when $(m=3,\ N=15,\ h=0.005,\ R=1)$ is applied. For n=1, 2 and 3, the NF decreases horizontally from 125.3000 to 3.6622, 50.9604 to 1.1678 and 25.5825 to 0.5546 by increasing the length as 10, 20, 30, 40 and 50. When circumferential wave numbers increase vertically from 1 to 5 then NF decreases horizontally from 125.3000 to 0.3228 by increasing the length as 10, 20, 30, 40 and 50.

Table 1
Variation in Natural Frequencies with Different Thickness against *n* for C-C Four Layered CS when $(m=2,\ N=10,\ L=20,\ R=1)$

n	***h*=0.001**	***h*=0.005**	***h*=0.007**	***h*=0.01**	***h*=0.05**
1	40.2391	40.2391	40.2391	40.2391	40.2390
2	13.8976	13.8975	13.8975	13.8974	14.0698
3	6.6687	6.6687	6.6692	6.6722	9.2179
4	3.8602	3.8615	3.8664	3.8886	12.8277
5	2.5044	2.5112	2.5323	2.6227	19.9483

Table 2
Variation in Natural Frequencies with Different Length against *n* for C-C Four Layered CS when $(m=3,\ N=15,\ h=0.005,\ R=1)$

n	***L*=10**	***L*=20**	***L*=30**	***L*=40**	***L*=50**
1	125.3000	40.2129	19.0408	10.9725	7.1042
2	50.9604	13.8885	6.2825	3.5563	2.2828
3	25.5825	6.6643	2.9860	1.6850	1.0808
4	15.0688	3.8590	1.7260	0.9774	0.6327
5	9.8595	2.5096	1.1319	0.6575	0.4472

Figures 1 and 2 illustrate the variation in natural frequencies against *n* for SS-SS four layered cylindrical shell. In figure 1, the natural frequency of SS-SS four layered CS is maximum at *N*=1 after that the natural frequency decreases with increasing the power exponent law and length against *n*. In figure 2, the natural frequency of SS-SS four layered cylindrical Shell is increased at *h*=10, then decreased at *h*=2 and *h*=3 and then continuously increased for various values of *h* against *n*.

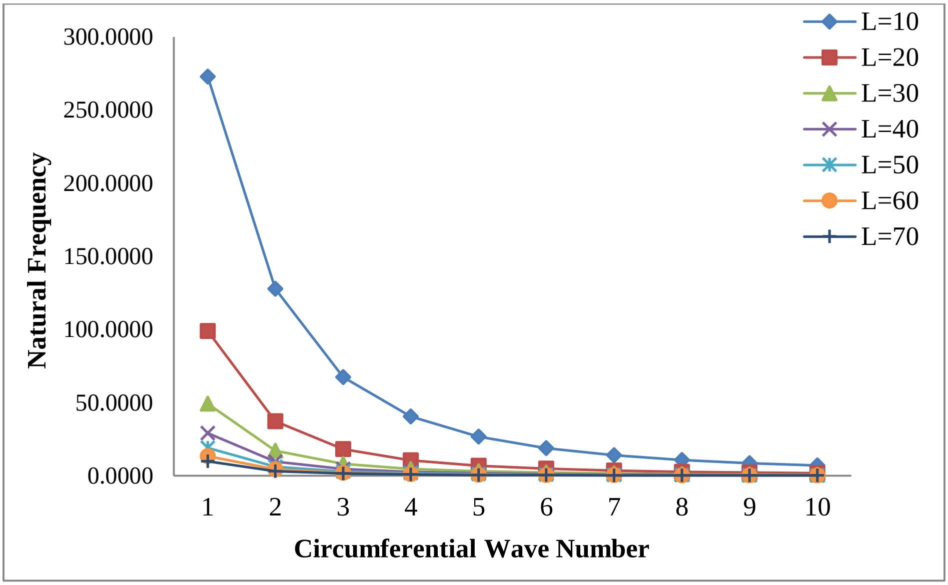

Figure 1: Variation in NFs with Different Length against *n* for SS-SS Four Layered CS when $(m=3,\ N=1,\ h=0.002,\ R=1)$

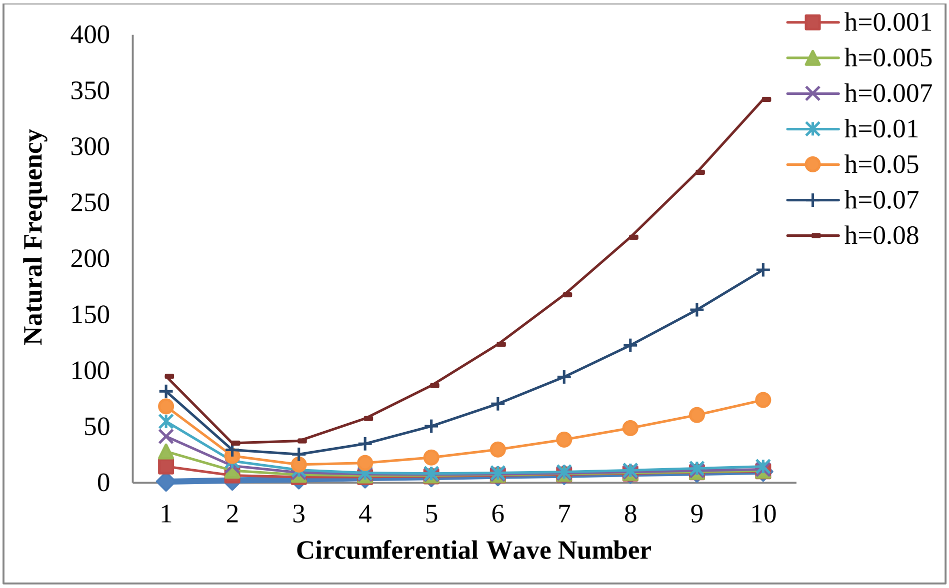

Figure 2: Variation in NFs with Different Thickness against *n* for SS-SS Four Layered CS when $\left(m=3,\ N=1,\ L=20,\ R=1\right)$

For authenticity of the current work, results for the C-C and SS-SS four layered CS are compared with others available in literature, as shown in tables 3 and 4. In which frequency parameters are compared with those presented in Iqbal *et al.* (2009) and Loy *et al.* (1999). In table 3, given the comparison of NFs against *n* for C-C CS when *(m*=1, *h*=0.002, *L*=20, *R*=1*)*. In three-layered CS, the behavior of the natural frequency against wave numbers at *N*=1 is unique, according to Iqbal *et al.* (2009). Like for all wave numbers, its behavior is decreasing. According to the present analysis, in a four layered cylindrical shell, the behavior of natural frequency against wave numbers is also unique, as at all wave numbers it decreases smoothly. In table 4, given the comparison of NFs against *n* for SS-SS CS with (*m*=1, *h*=0.002, *L*=20, *R*=1). In three-layered cylindrical shell, the behavior of natural frequency against wave numbers at *N*=1 and *N=2* has fluctuated, according to Loy *et al.* (1999). At some wave numbers, its behavior is increasing, while at others, it is decreasing. But according to the present analysis, in four-layered cylindrical shell, the behavior of natural frequency against wave numbers is unique, as at all wave numbers it decreases smoothly.

Table 3
Comparison of NFs with Power Exponent Law against *n* for C-C Cylindrical Shell when $\left(m=1,\ h=0.002,\ L=20,\ R=1\right)$

Wave Number	**Three Layered Iqbal *et al.* (2009)**	**Four Layered Present Analysis**
n	***N*=1**	***N*=1**
1	28.6132	40.6866
2	9.65474	14.0525
3	5.83010	6.7418
4	7.4354	3.9014
5	0	0

Table 4
Comparison of NFs with Different Power Exponent Law against n for SS-SS Cylindrical Shell when $\left(m=1,\ h=0.002,\ L=20,\ R=1\right)$

Wave Number	**Three Layered Loy *et al.* (1999)**		**Four Layered Present Analysis**	
n	***N*=1**	***N*=2**	***N*=1**	***N*=2**
1	13.211	13.103	13.4000	13.3372
2	4.480	4.4435	4.3557	4.3283
3	4.1569	4.1235	2.0613	2.0487
4	7.0384	6.9820	1.1869	1.1799
5	11.241	11.151	0.7680	0.7638

4. CONCLUSIONS

In the current study, four layered CS vibration frequency analysis is complete with different thickness, power exponent law and length of the shell’s layers for different edge conditions. For the displacement relations of strain and curvature, the Sander shell theory is used. Using the Rayleigh Ritz approach, the frequency equation of shell is found. For present Cs, results show the natural frequencies behavior with C-C and SS-SS boundary conditions. It observed that NFs are improved by increasing the thickness against circumferential wave number. By increasing the power exponent law and length against circumferential wave number, NFs are reduced. Thickness, length and power exponent law have an effect on NFs of the CS. Also, circumferential wave number has an effect on the NFs of the CS. Fs increased and decreased by circumferential wave number.

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EVALUATION OF LANDFILL SITES FOR URBAN AREAS BY USING INTERVAL-VALUED Q-RUNG ORTHOPAIR FUZZY MACLAURIN SYMMETRIC MEAN OPERATORS

Sana Fatima, Madiha Ghamkhar§, Khurrem Shehzad,
and **Muhammad Usman**
Department of Mathematics and Statistics
University of Agriculture, Faisalabad, Pakistan
§Email: madiha.ghamkhar@uaf.edu.pk

ABSTRACT

With the growth of the economy, the population in urban areas is growing. The amount of garbage generated in urban areas is increasing rapidly day by day. Proper disposal of garbage is an important issue for the global environment. In the municipal waste management system, the selection of landfill sites for solid waste disposal plays a core hearted role. The Maclaurin symmetric mean technique is a useful tool to accumulate information. It is capable of capturing the interconnection between several input arguments. We will evaluate landfill site selection using interval-valued q-rung orthopair fuzzy weighted Maclaurin symmetric mean operator and interval-valued q-rung orthopair fuzzy weighted dual Maclaurin symmetric mean operator. A comparative analysis will be presented in this study.

KEY WORDS

Multiple attribute group decision making, interval-valued q-rung orthopair fuzzy weighted dual MSM ($IV\,q{-}ROFWDMSM$) operator, interval-valued q-rung orthopair fuzzy weighted MSM ($IV\,q - ROFWMSM$) operator.

1. INTRODUCTION

For human health and the environment "garbage siege" is a crucial issue. The urban population has grown exponentially with economic development. Disposal of solid waste is very important. Landfilling is still an integral part of the solid waste management system (Ren, Yuan and Lin, 2021). The fuzzy set theory, pioneered by Zadeh (1965), handles hazy and ambiguous circumstances. Fuzzy sets have made significant contributions to virtually every scientific subject. It has practical and theoretical applications in computer science, medical sciences, engineering, arts, humanities, energy, materials, economics, and medicine (Kahraman, Öztayşi and Çevik Onar (2016).

Atanassov (1986 & 1989 introduced Intuitionistic fuzzy sets as described by satisfying degree and non- satisfying degree with the limit that their summation should be bounded between 0 and 1. Yager developed a new notion known as the Pythagorean fuzzy set. PFS is defined as the summation of the squares of the satisfying and non-satisfying degrees if they are equal or less than one. PFS is an extended form of IFS (Xu et al. 2019). Yager (2016) developed a new technique called q-ROFS to solve inadequacies of PFS. The addition of q-th power of positive and non-positive degrees in q-ROFS must be between 0 and 1. q-ROFS is an extended form of IFS and PFS. Several techniques, including the q-ROFHM technique, q-ROFBM technique, q-ROFMSM technique, q-ROFPMSM technique, and q-ROFMM technique, have been established (Xu et al., 2019).

Express opinions by an interval number, subinterval of [0,1], are flexible for experts. The IVq-ROFSs empower DMs to evaluate their positive and negative levels to a set of options using an interval value. q-RIVOFS is refinement of q-ROFS (Joshi et al., 2018). The difficulty of selecting the best option from a predetermined number of choices based on a set of criteria provided by a group of experts with a high level of collective experience on these specific criteria is addressed by MAGDM Yang et al. (2015). Aggregation operators are used to combining experts' diverse assigned values to an attribute into a single evaluation value for each alternative. Maclaurin created the MSM operator, which has an edge over ordinary operators in that it can investigate the connection between several input arguments (Wei et al., 2019).

The remaining paper is structured as follows. Section 2 consists definition of $IV\ q - ROFSs$ and MSM operator. Section 3 recalls $IV\ q - ROFWMSM$ and $IV\ q - ROFWDMSM$ operators. Section 4 applies the approach in the evaluation of landfill sites for urban areas. Section 5 summarizes the study.

2. PRELIMINARIES

Interval-Valued q-Rung Orthopair Fuzzy Set

Definition 1: (Joshi et al., 2018)

Assume V be a set. A IVq-ROFS is an object having the form

$$\tilde{G} = \left\{\left\langle v, \left(\tilde{\zeta}_{\tilde{G}}(v), \tilde{\xi}_{\tilde{G}}(v)\right)\right\rangle \middle| v \in V\right\}, \tag{1}$$

where $\tilde{\zeta}_{\tilde{G}}(v) \subset [0,1]$ and $\tilde{\xi}_{\tilde{G}}(v) \subset [0,1]$ are interval numbers, $\tilde{\zeta}_{\tilde{G}}(v) = [\lambda(v), \delta(v)]$, $\tilde{\xi}_{\tilde{G}}(v) = [ø(v), \rho(v)]$ with the restriction $0 \le (\delta(v))^q + (\rho(v))^q \le 1, \forall v \in V, q \ge 1$. Let $\tilde{g} = ([\lambda, \delta], [ø, \rho])$ be a IVq-ROFN, $S(\tilde{g}) = 1/4[(2 + \lambda^q + \delta^q - ø^q - \rho^q)]$ and $H(\tilde{g}) = (\lambda^q + \delta^q - ø^q - \rho^q)/2$ are the score and accuracy functions of a IVq-ROFN $\tilde{g}$.

2.1 The Maclaurin Symmetric Mean Operator

Maclaurin (1729) familiarized this operator. In modern information fusion theory, MSM used as a basic aggregation operator to accumulate numerical values.

Definition 2: (Maclaurin, 1729)

Let $\tilde{g}_r (r = 1,2, \dots, m)$ be a set of non-negative real numbers, and $t = (1,2, \dots, m)$. If

$$MSM^{(t)}(\tilde{g}_1, \tilde{g}_2, \dots, \tilde{g}_m) = \left(\frac{\sum_{1 \le r_1 < r_2 < \cdots < r_t \le m} \prod_{s=1}^{t} \tilde{g}_{r_s}}{C_m^t}\right)^{\frac{1}{t}} \tag{2}$$

$MSM^{(t)}$, where $(r_1, r_2, ..., r_t)$ covers all t-tuple combinations of $(1,2, \dots, m)$, and C_m^t is the binomial coefficient, became known as the MSM operator.

3. SOME MSM OPERATOR WITH IVq-ROFNS

a) The IVq-ROFWMSM operator

Definition 3: (Wang et al., 2019).

Let $\tilde{g}_s = ([\lambda_s, \delta_s], [\varnothing_s, \rho_s])(s = 1, 2, \ldots, m)$ be a set of IVq-ROFNs and their weight vector be $v_r = (v_1, v_2, \ldots, v_m)^T$, thereby satisfying $v_r \in [0,1]$ and $\sum_{r=1}^{m} v_r = 1$.Then IVq-ROFWMSM operator is defined as:

$$IVq-ROFWWMSM^{(t)}(\tilde{g}_1, \tilde{g}_2, \ldots, \tilde{g}_m) = \left(\frac{\sum_{1\le r_1<r_2<\cdots<r_t\le m}\left(\prod_{s=1}^{t}\left(\tilde{g}_{r_s}\right)^{v_{r_s}}\right)}{C_m^t}\right)^{\frac{1}{t}}$$

$$= \left\{ \begin{array}{l} \left[\left(\sqrt[q]{1 - \prod_{1\le r_1<r_2<\cdots<r_t\le m} \left(1 - \left(\prod_{s=1}^{t} (\lambda_s)^{v_{r_s}}\right)^{q}\right)^{\frac{1}{C_m^t}}} \right)^{\frac{1}{t}}, \left(\sqrt[q]{1 - \prod_{1\le r_1<r_2<\cdots<r_t\le m} \left(1 - \left(\prod_{s=1}^{t} (\delta_s)^{v_{r_s}}\right)^{q}\right)^{\frac{1}{C_m^t}}} \right)^{\frac{1}{t}} \right] \\ \left[\sqrt[q]{1 - \left(1 - \left(\prod_{1\le r_1<r_2<\cdots<r_t\le m} \left(1 - \prod_{s=1}^{t} \left(1 - (\varnothing_s)^q\right)^{v_{r_s}}\right)\right)^{\frac{1}{C_m^t}}\right)^{\frac{1}{t}}}, \sqrt[q]{1 - \left(1 - \left(\prod_{1\le r_1<r_2<\cdots<r_t\le m} \left(1 - \prod_{s=1}^{t} \left(1 - (\rho_s)^q\right)^{v_{r_s}}\right)\right)^{\frac{1}{C_m^t}}\right)^{\frac{1}{t}}} \right] \end{array} \right\} \quad (3)$$

b) The IVq-ROFWDMSM operator

Attribute weights must be taken into account in real-world MADM issues.

Definition 4: (Wang et al., 2019)

Let $\tilde{g}_s = ([\lambda_s, \delta_s], [\varnothing_s, \rho_s])(s = 1, 2, \ldots, m)$ be a set of $q - RIV\ OFNs$ and their weight vector be $v_r = (v_1, v_2, \ldots, v_m)^T$ thereby satisfying $v_r \in [0,1]$ and $\sum_{r=1}^{m} v_r = 1$. If

$$IVq-ROFWDMSM^{(t)}(\tilde{g}_1,\tilde{g}_2,\dots,\tilde{g}_m)=\frac{1}{t}\left(\prod_{1\le r_1<r_2<\cdots<r_t\le m}\left(\sum_{s=1}^{t}v_{r_s}\tilde{g}_{r_s}\right)^{\frac{1}{C_m^t}}\right)$$

$$=\left\{\begin{array}{l}\left[\sqrt[q]{1-\left(1-\prod_{1\le r_1<r_2<\cdots<r_t\le m}\left(1-\prod_{s=1}^{t}\left(1-(\lambda_s)^q\right)^{v_{r_s}}\right)^{\frac{1}{C_m^t}}\right)^{\frac{1}{t}}},\sqrt[q]{1-\left(1-\prod_{1\le r_1<r_2<\cdots<r_t\le m}\left(1-\prod_{s=1}^{t}\left(1-(\delta_s)^q\right)^{v_{r_s}}\right)^{\frac{1}{C_m^t}}\right)^{\frac{1}{t}}}\right], \\ \left[\left(\sqrt[q]{1-\prod_{1\le r_1<r_2<\cdots<r_t\le m}\left(1-\left(\prod_{s=1}^{t}(\emptyset_s)^{v_{r_s}}\right)^q\right)^{\frac{1}{C_m^t}}}\right)^{\frac{1}{t}},\left(\sqrt[q]{1-\prod_{1\le r_1<r_2<\cdots<r_t\le m}\left(1-\left(\prod_{s=1}^{t}(\rho_s)^{v_{r_s}}\right)^q\right)^{\frac{1}{C_m^t}}}\right)^{\frac{1}{t}}\right]\end{array}\right\} \quad (4)$$

4. EVALUATION OF LANDFILL SITES FOR URBAN AREAS

4.1 Numerical Example

The demand for industrial commodities has grown with the rapid growth of the urban population. As a result, the amount of waste generated per day is increasing at an alarming rate. Landfills are built to compact garbage into compacted layers to decrease volume. The selection of landfill sites plays a key role to get at most advantages and minimizing the harmful effects. A landfill is a waste disposal place that is designed to protect and preserve the environment. To construct new landfill there are several most important factors must be considered as geographical location, operating cost, traffic conditions and environmental pollution. All these factors vary from site to site. To choose the best one, evaluation of sites is necessary.

4.2 Evaluation Process of Landfill Sites for Urban Areas

There are four addresses G_r (r = 1, 2, 3, 4) as candidate sites for landfill construction. The specialists ($\widetilde{D}_1$, $\widetilde{D}_2$, $\widetilde{D}_3$) with weight vector v = $(0.3, 0.3, 0.4)^T$ will evaluate the candidate sites based on four indicators(attributes) as (A_1) geographical location, (A_2) operating cost, (A_3) traffic condition and (A_4) environmental pollution. The four possible contestant sites G_r (r = 1, 2, 3, 4) are to be assessed by the experts with the above four characteristics (whose weight vector v = $(0.40, 0.25, 0.20, 0.15)^T$.

Step 1: To obtain the entire *IV q − ROFNs* g_{rs}^{α} (r = 1, 2, 3, 4, s = 1, 2, 3, 4) of G_r, accumulate all *IV q − ROFNs* g_{rs}^{β} by applying the *IV q – ROFWA* (*IV q − ROFWG*) operator. Table 4 (with q = 4) displays the results.

Table 1
IVq-ROFN Decision Matrix (E_1)

	A_1	A_2	A_3	A_4
G_1	([0.7,0.8],[0.3,0.6])	([0.6,0.8],[0.4,0.5])	([0.7,0.9],[0.5,0.6])	([0.6,0.7],[0.4,0.7])
G_2	([0.7,0.9],[0.2,0.3])	([0.6,0.9],[0.2,0.5])	([0.8,0.9],[0.4,0.6])	([0.6,0.8],[0.2,0.4])
G_3	([0.5,0.7],[0.3,0.4])	([0.6,0.8],[0.4,0.5])	([0.6,0.7],[0.1,0.2])	([0.7,0.8],[0.1,0.3])
G_4	([0.7,0.9],[0.1,0.2])	([0.6,0.8],[0.3,0.5])	([0.5,0.6],[0.4,0.5])	([0.7,0.9],[0.3,0.5])

Table 2
IVq-ROFN Decision Matrix (E_2)

	A_1	A_2	A_3	A_4
G_1	([0.6,0.7],[0.2,0.4])	([0.8,0.9],[0.3,0.6])	([0.6,0.9],[0.1,0.3])	([0.6,0.7],[0.3,0.4])
G_2	([0.8,0.9],[0.1,0.2])	([0.7,0.8],[0.5,0.6])	([0.6,0.7],[0.2,0.4])	([0.7,0.9],[0.1,0.3])
G_3	([0.8,0.9],[0.3,0.5])	([0.7,0.9],[0.3,0.4])	([0.5,0.8],[0.2,0.4])	([0.6,0.8],[0.2,0.3])
G_4	([0.6,0.8],[0.3,0.4])	([0.7,0.9],[0.1,0.2])	([0.6,0.7],[0.2,0.3])	([0.3,0.5],[0.6,0.7])

Table 3
IVq-ROFN Decision Matrix (E_3)

	A_1	A_2	A_3	A_4
G_1	([0.5,0.7],[0.1,0.2])	([0.5,0.6],[0.2,0.4])	([0.2,0.4],[0.3,0.4])	([0.2,0.4],[0.3,0.5])
G_2	([0.5,0.7],[0.1,0.2])	([0.4,0.5],[0.2,0.4])	([0.5,0.7],[0.3,0.4])	([0.4,0.5],[0.2,0.3])
G_3	([0.1,0.2],[0.5,0.8])	([0.7,0.8],[0.4,0.5])	([0.6,0.8],[0.2,0.4])	([0.1,0.2],[0.6,0.8])
G_4	([0.2,0.3],[0.5,0.6])	([0.3,0.4],[0.6,0.8])	([0.4,0.5],[0.5,0.7])	([0.6,0.8],[0.2,0.5])

Table 4
The Aggregated Results by the $IV\, q - ROFWA$ Operator

	A_1	A_2	A_3	A_4
G_1	([0.6018,0.7374], [0.1712,0.3424])	([0.6703,0.8019], [0.2718,0.4830])	([0.5851,0.8317], [0.2515,0.4144])	([0.5327,0.6330], [0.3270,0.5173])
G_2	([0.6954,0.8523], [0.1231,0.2259])	([0.5954,0.7933], [0.2633,0.4830])	([0.6703,0.7957], [0.2896,0.4517])	([0.5954,0.7933], [0.1625,0.3270])
G_3	([0.6350,0.7590], [0.3680,0.5643])	([0.6757,0.8401], [0.3669,0.4676])	([0.5759,0.7763], [0.1625,0.3249])	([0.5845,0.7219], [0.2521,0.4441])
G_4	([0.5851,0.7863], [0.2647,0.3821])	([0.5880,0.7885], [0.2847,0.4584])	([0.5123,0.6107], [0.3552,0.4908])	([0.6015,0.8066], [0.3140,0.5531])

Table 5
The Aggregated Results by the $IV\, q$ – $ROFWMSM$ ($IV\, q - ROFWDMSM$) Operator

	$IV\, q$ – $ROFWMSM$	$IV\, q - ROFWDMSM$
G_1	([0.8861,0.9615],[0.1841,0.3083])	([0.4325,0.5601],[0.7297,0.8209])
G_2	([0.8994,0.9511],[0.1596,0.2712])	([0.4639,0.6024],[0.7016,0.7947])
G_3	([0.8919,0.9409],[0.2247,0.3352])	([0.4465,0.5722],[0.7438,0.8284])
G_4	([0.8742,0.9331],[0.2141,0.3382])	([0.4068,0.5555],[0.7566,0.8334])

Step 2: The fused values for k=2 are given in Table 5 and are obtained by fusing all IV q − ROFNs together by IV q − ROFWMSM (IV q −ROFWDMSM).

Step 3. The scores of the alternatives are displayed in Table 6 based on the fused values in Table 5 and the scoring functions of IVq-ROFNs.

Step 4. Use the values from Table 6 to rank all of the choices, and the results are given in Table 7.

G_2 is, without a doubt, the best option.

Table 6
The Scores of Alternatives

	$IV\, q$ – $ROFWMSM$	$IV\, q - ROFWDMSM$
A_1	([0.8861,0.9615],[0.1841,0.3083])	([0.4325,0.5601],[0.7297,0.8209])
A_2	([0.8994,0.9511],[0.1596,0.2712])	([0.4639,0.6024],[0.7016,0.7947])
A_3	([0.8919,0.9409],[0.2247,0.3352])	([0.4465,0.5722],[0.7438,0.8284])
A_4	([0.8742,0.9331],[0.2141,0.3382])	([0.4068,0.5555],[0.7566,0.8334])

Table 7
The Ordering of Alternatives

	Ordering
IVq−ROFWMSM Operator	$G_2 > G_1 > G_3 > G_4$
IVq−ROFWDMSM Operator	$G_2 > G_1 > G_3 > G_4$

4.3 Comparative Analysis

The IVq-ROFWMSM and IVq-ROFWDMSM operators are now compared to the IVq-ROFWA and IVq-ROFWG operators. Table 8 displays the results.

Table 8
Comparative results

	Ordering
IVq−ROFWA Operator	$G_2 > G_3 > G_1 > G_4$
IVq−ROFWG Operator	$G_2 > G_3 > G_1 > G_4$

CONCLUSION

The MSM operator can handle MAGDM problems with IVq-ROF information and can take into account the interrelationships between arguments. Based on IVq-ROFWMSM and IVq-ROFWDMSM operators, we evaluate the landfill sites selection for urban areas, which helps the decision makers to choose the best platform. In the future, we will investigate more MAGDM methods under IV-qROFSs situations and study further real-life applications in many other fields under uncertain environment.

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A NEW INTEGRAL TRANSFORM "ALI AND ZAFAR" TRANSFORMATION AND ITS APPLICATIONS IN NUCLEAR PHYSICS

Ali Moazzam and **Muhammad Zafar Iqbal**
Department of Mathematics and Statistics
University of Agriculture, Faisalabad, Pakistan
Email: alimoazzam7309723@gmail.com
mzts2004@hotmail.com

ABSTRACT

Integral transform technique is advantageous tool for precise calculations in mathematics, as it converts complicated problematic phenomena into a simple one. It is tranquil to distinct the properties of a specified function from the parental function, after integration has been consigned. In this article, we are going to introduce a new integral transforms technique will know as Ali and Zafar transform (AZ-transform), which is applied to find solution of ordinary linear differential equations with logarithmic functions as coefficients of the variable value and its presentations in different era of engineering, the radioactive decay problems take place in the field of nuclear physics. Descriptive properties and examples appear to show the efficiency of its suitability in solving differential equations.

KEYWORDS

Ali and Zafar Transformation, Logarithmic Functions, Radio Active Decay.

1. INTRODUCTION

Difference equalities characterize an important role in demonstrating many problems in the technical arenas of science, its significance led to continue offer of novel approaches to resolve equalities including higher derivatives in simplest, quick and utmost effectual means (Herve and Lucquin, 2016). Demonstration of integral techniques to solve many problems in the junction grit in the structure of many significant technical and engineering areas of study, such as atomic decay problems accrue in nuclear physics by (Emad *et al.*, 2020), heat behavior differential equation by (Mehdi and Dehghan, 2009, Elaf *et al.*, 2018), computing and communications by (Guepreet and Arora, 2016) and several additional fields of science, where differential equalities are placed to represent the variations that might occur on the base of these conditions occur in phenomena all the way through the application of altered circumstances.

The offered Ali and Zafar integral technique of transform has presented to enable the procedure of resolving differential partial and ordinary equations in this period, it has generated from integral Fourier transform and Laplace technique (Aproova *et al.*, 2019), and possess all integral Fourier and Laplace central properties through profounder connection with integral Elzaki technique (Tarig and Elzaki, 2012) and Integral Mohand, Laplace technique, integral Abood transforms technique (Sudhanshu and Chauhan, 2019).

Although Ali and Zafar integral technique to transform function is illustrated from already well known techniques, it supports to ease and straight-forward-ness in which offers it an advance technique over other integral methods to resolving differential ordinary and partial equations.

Decay, decompose, rot, putrefy, and ruin all refer to the process of decomposition. The term "decay" refers to a gradual deterioration from a position of soundness or perfection, as in a crumbling mansion. When applied to organic stuff, decompose stresses a chemical breakdown and corruption.

Exponential decay is linked to the process of radioactive decay and population decline. The half-life of a substance is the amount of time it proceeds for half of it to deteriorate or disintegrate.

The emitting radiation of the thorium and uranium having natural rudiments is present the basics in the atomic-physics. Emission of radiation might occur as expected or prompted by non-natural approaches (Jean *et al.,* 2005). There are numerous kinds of atomic degenerations that be contingent on the kind of emanated particle emission like as α-radiations, γ- radiations, β- radiations and additional kinds as demonstrations in fig. 1. (Walter *et al.,* 2017).

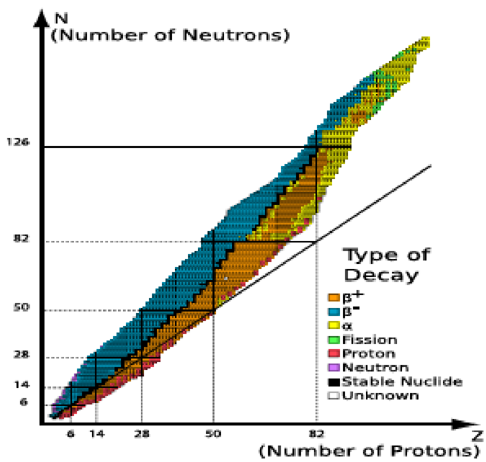

Figure 1: Nuclear Decay of Isotopes

Ali and Zafar Transformation

The recommended integral transform is expressed for a logarithmical order task. The kernel of “Ali and Zafar” transformation is symbolized by AZ(.) is described as integral equation,

$$AZ\left[f(ln\tau)\right] = \int_1^\infty \tau^{-\left(\frac{1}{s^2}+1\right)} f\left(ln\tau\right) d\tau \tag{1}$$

where $ln\,\tau \geq 1,$ and $"s"$ is transformed variable. The above integral is convergent in given interval.

Inverse of Transformation

If $F(s)$ is a transformed function after applying transformation $AZ\left[f(ln\tau)\right]$, then inverse of AZ transform can be defined as,

$$AZ^{-1}\left[F(s)\right] = f(ln\tau)\,. \tag{2}$$

2. MAIN RESULTS

Applying Ali and Zafar Integral Technique into Some Important Functions

In **"Ali and Zafar"** integral transform technique, the survival circumstance essential be met to put the transformation in function whose dependency upon logarithmic functions.

The presence of $AZ\left[f(ln\tau)\right]$ to any task $f(ln\tau)$ is for $\tau \geq 1$, need to be piece-wise continuous and in logarithmic form.

"AZ" transform technique can be applied to transform the task given below.

Table 2
Transformation for Some Functions

S#	Function $f(ln\tau)$	Transformed Function F(*S*) and Their Convergence
1.	1	s^2
2.	τ	$s^2/1-s^2, \quad s>0$
3	$\tau^n, \quad n=1,2,3,......$	$s^2/1-ns^2, \quad s\geq 0$
4.	$(ln\tau)^n$	$n!\left(s^2\right)^{n+1}$
5.	$\tau^n\left(ln\tau\right)$	$\left(s^2\right)^2/\left(1-ns^2\right)^2$
6.	$sin(b\left(ln\tau\right)), \quad \tau\geq 0$	$b\left(s^2\right)^2/1+\left(bs^2\right)^2, \quad s>0$
7.	$cos(b\left(ln\tau\right)), \quad \tau\geq 0$	$s^2/1+\left(bs^2\right)^2, \quad s>0$
8.	$sinh(b\left(ln\tau\right)), \quad \tau\geq 0$	$b\left(s^2\right)^2/1-\left(s^2b\right)^2, \quad s>\lvert b\rvert$
9.	$cosh(b\left(ln\tau\right)), \quad \tau\geq 0$	$s^2/1-\left(s^2b\right)^2, \quad s>0$

Theorem (Times Derivative)

Let *F(s)* be the "AZ" transform of $f(ln\tau)$ then,

$$AZ\left[f'(ln\tau)\right] = -f\left(0\right) + \frac{1}{s^2}F(s) \tag{3}$$

$$AZ\left[f''(ln\tau)\right] = -f'(0) - \frac{1}{s^2} f(0) + \frac{1}{\left(s^2\right)^2} F(s) \tag{4}$$

$$AZ\left[f'''(ln\tau)\right] = -f''(0) - \frac{1}{s^2} f'(0) - \frac{1}{\left(s^2\right)^2} f(0) + \frac{1}{\left(s^2\right)^3} F(s) \tag{5}$$

$$AZ\left[f^{(n)}(ln\tau)\right] = \sum_{i=1}^{n} \frac{(-1)^{2i+1} f^{(i-1)}(0)}{\left(s^2\right)^{n-i}} + \frac{1}{\left(s^2\right)^n} F(s) \tag{6}$$

Proof:

1. $AZ\left[f'(ln\tau)\right] = \int_1^\infty \tau^{-\left(\frac{1}{s^2}+1\right)} f'(ln\tau)\, d\tau$

$$= \tau^{-\frac{1}{s^2}} \int_1^\infty \frac{f'(ln\tau)}{\tau} d\tau + \frac{1}{s^2} \int_1^\infty \tau^{-\left(\frac{1}{s^2}+1\right)} f(ln\tau)\, d\tau$$

$$= \tau^{-\frac{1}{s^2}} f(ln\tau)\Big|_1^\infty + \frac{1}{s^2} F(s) = -f(0) + \frac{1}{s^2} F(s)$$

2. $AZ\left[f''(ln\tau)\right] = \int_1^\infty \tau^{-\left(\frac{1}{s^2}+1\right)} f''(ln\tau)\, d\tau$

$$= \tau^{-\frac{1}{s^2}} \int_1^\infty \frac{f''(ln\tau)}{\tau} d\tau + \frac{1}{s^2} \int_1^\infty \tau^{-\left(\frac{1}{s^2}+1\right)} f'(ln\tau)\, d\tau$$

$$= \tau^{-\frac{1}{s^2}} f'(ln\tau)\Big|_1^\infty + \frac{1}{s^2} F'(s)$$

By using (2), $= -f'(0) - \frac{1}{s^2} f(0) + \frac{1}{\left(s^2\right)^2} F(s)$

3. $AZ\left[f'''(ln\tau)\right] = \int_1^\infty \tau^{-\left(\frac{1}{s^2}+1\right)} f'''(ln\tau)\, d\tau$

$$= \tau^{-\frac{1}{s^2}} \int_1^\infty \frac{f'''(ln\tau)}{\tau} d\tau + \frac{1}{s^2} \int_1^\infty \tau^{-\left(\frac{1}{s^2}+1\right)} f''(ln\tau)\, d\tau$$

$$= \tau^{-\frac{1}{s^2}} f''(ln\tau)\Big|_1^\infty + \frac{1}{s^2} F''(s)$$

By using (3), $= -f''(0) - \frac{1}{s^2} f'(0) - \frac{1}{\left(s^2\right)^2} f''(0) + \frac{1}{\left(s^2\right)^3} F(s)$

4. Can be proved by using mathematical induction.

3. APPLICATIONS

Applying of "Ali and Zafar" Transformation into Differential Ordinary Equations

Linear coordination that ruled by difference equalities, Ali and Zafar transform technique can be applied as an effectual approach to examine their basic appearances in comeback into original data. The effectiveness of AZ technique of transformation could be seen in resolving several certain problems at initial condition defined by differential ordinary equations.

Application of "AZ" transform into ordinary differential first-order equations in nuclear physics

With the use of ordinary linear differential first order equations, (Aggarwal *et al.,* 2018) mathematically described the decay difficulties of a substance.

$$\frac{dN}{dt} = -\sigma N \tag{7}$$

with initial condition as

$$N(t_0) = N_0, \tag{8}$$

where σ is a real positive number, N is the quantity of atoms in an component at t-time and N_0 is the initial the number of atoms in an element at initial time t_0.

Sol: Let

$$N = N(ln\tau) \tag{9}$$

$$\frac{dN\left(ln\tau\right)}{d\tau} + \sigma N\left(ln\tau\right) = 0$$

By using (3)

$$-N\left(0\right) + \frac{1}{s^2}N(s) + \sigma N(s) = 0\ ,\ \left[\frac{1+\sigma s^2}{s^2}\right]N(s) = N_0$$

$$N(s) = N_0\left[\frac{s^2}{1+\sigma s^2}\right] \tag{10}$$

By taking inverse of (10), we will get, $N(ln\tau) = N_0\tau^{-\sigma}$

Again by using (9),

$$N(\tau) = N_0 e^{-\sigma\tau} \tag{11}$$

4. CONCLUSION

Ali and Zafar alteration is proficient of resolving differential ordinary equations and functions, one or the other by means of its (table 1) or by means of its characterization and propositions. The preceding calculation showed the aptitude of Ali and Zafar transformation technique of provided that a novel and basic universal technique to resolve the emitting radiation decays difference ordinary equation. So it can be decided that Ali and Zafar transformation technique is much effective to solve differential ordinary equations.

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BIO-CONVECTION IMPACT ON NANOFLUID FLOW OVER A STRETCHING SURFACE

Faisal Fareed Bukhari, Madiha Ghamkhar[§], Ghulam Fareed and **Muhammad Kashif**
Department of Mathematics and Statistics
University of Agriculture, Faisalabad, Pakistan
[§]Email: madiha.ghamkhar@uaf.edu.pk

ABSTRACT

Commercial cooling systems, microelectromechanical systems, innovative conditioning devices, and high-temperature monitoring devices that use evaporator coils and heat transfers all use nanofluids. The effects of bioconvection structured by a stretched surface in a magnetized 3-D flow of nanofluid are investigated in that work. The effects of nonlinear heat radiation on three-dimensional flow phenomena in a porous material are also investigated in that research. System of equations in the form of partial differential equations is transformed into the ordinary differential equation by using similarities transformations. Higher-order differential equations are converted into 1st order differential equations by using the shooting method. Graphical results for different parameters on velocity, temperature, and concentration profiles are discussed graphically by using the MAT-Lab tool, named bvp4c. An increase in the Buoyancy ratio parameter increases the concentration of nanoparticles. Rather concentration of nanoparticles is decreased by increasing the thermophoresis parameter.

1. INTRODUCTION

Nanofluids play an important role in commercial cooling systems, microelectromechanical systems, innovative conditioning devices, and high-temperature monitoring devices using evaporator coils as well as heat transfers. In several sectors, like biology, chemistry, physics, and nuclear engineering, nanomaterials are extremely relevant, especially for natural occurrences. A fluid containing the nanometer (1 to 100 nm)-sized particles in the base fluid is referred to as nanofluid. It has many physical properties that make the potential benefits for heat transfer analysis. Nanoparticles, which are typically used in nanofluids, are composed of metals, oxide, and carbon nanotubes carbide. The base fluid used in nanoparticles can be water, ethylene glycol, and oil. The magnetohydrodynamic three-dimensional stream of nanofluid over a convectively heated nonlinear stretching surface was explored by Hayat et al. [2012]. Raju et al. [2016] investigated dual solutions for three-dimensional MHD flow of a nanofluid via a nonlinearly permeable stretching sheet. Hayat et al. [2016] investigated the effects of melting heat and internal heat generation in a nonlinear stretching surface with changing thickness in a stagnation point flow of Jeffrey fluid. Gireesha et al. [2017] studied the MHD three-dimensional double-diffusive flow of Casson nanofluid over a stretching surface with buoyancy forces and nonlinear thermal radiation. The properties of the exponential space-dependent heat source and cross-diffusion effects in Marangoni convective heat mass transfer flow due to an infinite disc. The three-dimensional flow of a Jeffery fluid over a linearly stretched sheet was studied by Hayat et al. [2018]. Waqas et al. [2021] have modeled the mixed convection flow of Jeffrey nanomaterial.

2. MATHEMATICAL METHOD

In a time-dependent mixed convective boundary layer flow, gyrotactic microorganisms travel around a rotating sheet in Jeffrey nanofluid. The velocity components along the x, y, and z axes are represented by u, v, and w, correspondingly. Stream is affected by the non-uniform heat source and magnetic field. By keeping the temperature T_w as a dependent of a sheet, suppose the mass flux of the Nanoparticle close to the surface is zero, T_∞ is climate values of temperature, C_∞ is Nano-particle volume fraction. Under standard boundary constraints, the conditions governing three-dimensional flow with heat and mass transfer are as follows:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \tag{1}$$

$$\begin{aligned} u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=\frac{\nu_f}{1+K_1}\Bigg[\frac{\partial^2 u}{\partial z^2}+\lambda\Bigg\{\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial x\partial z}+\frac{\partial v}{\partial z}\frac{\partial^2 u}{\partial y\partial z}+\frac{\partial w}{\partial z}\frac{\partial^2 u}{\partial z^2} \\ +u\frac{\partial^3 u}{\partial x\partial z^2}+v\frac{\partial^3 u}{\partial y\partial z^2}+w\frac{\partial^3 u}{\partial z^3}+u\frac{\partial^3 u}{\partial x\partial z^2}+v\frac{\partial^3 u}{\partial y\partial z^2}+w\frac{\partial^3 u}{\partial z^3}\Bigg\}\Bigg] \\ -\frac{\sigma B_0^{\,2}}{\rho_f}u-\frac{v}{l_1}u+\frac{1}{\rho_f}\Big[\left(1-C_f\right)\rho_f\beta g\left(T-T_\infty\right)-\left(\rho_p-\rho_f\right)g \\ +\left(C-C_\infty\right)-\left(n-n_\infty\right)g\gamma^*\left(\rho_m-\rho_f\right)\Big]. \end{aligned} \tag{2}$$

$$\begin{aligned} u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+w\frac{\partial T}{\partial z}=\alpha\frac{\partial^2 T}{\partial z^2}-\frac{1}{\left(\rho c\right)_f}\frac{\partial q_r}{\partial z} \\ +\frac{\left(\rho c\right)_p}{\left(\rho c\right)_f}\left(D_B\left(\frac{\partial T}{\partial z}\frac{\partial C}{\partial z}\right)+\frac{D_T}{T_\infty}\left(\frac{\partial T}{\partial z}\right)^2\right) \\ +Q_E\left(T_w-T_\infty\right)\exp\left[-\left(\frac{b}{\nu_f}\right)^{0.5}\left(1-\alpha t\right)^{-0.5}ny\right]. \end{aligned} \tag{3}$$

$$u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}+w\frac{\partial C}{\partial z}=D_B\frac{\partial^2 C}{\partial z^2}+\frac{D_T}{T\infty}\frac{\partial^2 T}{\partial z^2} \tag{4}$$

$$u\frac{\partial N}{\partial x}+v\frac{\partial N}{\partial y}+w\frac{\partial N}{\partial z}+\frac{bW_c}{\Delta C}\left[\frac{\partial}{\partial z}\left(N\frac{\partial C}{\partial z}\right)\right]=D_M\left(\frac{\partial^2 N}{\partial z^2}\right) \tag{5}$$

Along with boundary conditions

$$u = U_w \ , v = V_w, \ w = 0, \ k\frac{\partial T}{\partial y} = \alpha_f \left(T - T_\infty\right), \ N = N_w, D_B \frac{\partial C}{\partial z}$$
$$+\frac{D_T}{T_\infty}\frac{\partial T}{\partial z} = 0 \ \ for \ z = 0, u \to 0, v \to 0, \frac{\partial u}{\partial z} \to 0, \frac{\partial v}{\partial z} \to 0, \tag{6}$$
$$C \to C_\infty, T \to T_\infty, \ N \to N_\infty also \ for \ z \to \infty.$$

Here M, K, $K_{1,}$ and K_2 are magnetic field parameters, porosity parameters, the ratio of relaxation times to retardation times, and Deborah number respectively. Mixed convection parameter, buoyancy ratio constant and bioconvection Rayleigh number are denoted as $\lambda, Nr \, and \, Rb$ respectively. These parameters are defined as,

$$a = B_0\left(x + y\right)^{1-n}, \ l_o = l_1\left(x + y\right)^{1-n}, \ K = \frac{1}{al_0}, \ K_2 = \lambda B,$$
$$Nr = \left(\frac{\left(\rho_p - \rho_f\right)\left(C_w - C_\infty\right)}{\rho_f\left(1 - C_\infty\right)T_\infty\beta}\right)$$
$$\lambda = \left(\frac{\beta\gamma^*\left(1 - C_\infty\right)\left(T_w - T_\infty\right)}{au_w}\right), \ Rb = \left(\frac{\gamma^*\left(\rho_m - \rho_f\right)\left(n_w - n_\infty\right)}{\beta\rho_f\left(1 - C_\infty\right)T_\infty}\right), \ M = \frac{\sigma B_0^2}{\rho a}.$$

By using the values of the following similarity variables,

$$\left.\begin{aligned}
&u = a(\mathrm{x+y})^2 f'(\eta), \quad v = a(\mathrm{x+y})^2 g'(\eta), \quad \eta = \sqrt{\frac{a}{\nu_f}}\left(\mathrm{x+y}\right)^{\frac{n-1}{2}} z, \\
&T = T_\infty + \left(T_w - T_\infty\right)\theta\left(\eta\right), C = C_\infty + C_\infty\,\phi\left(\eta\right), \ N = N_\infty + \left(N_w - N_\infty\right)\chi\left(\eta\right) \\
&w = -\sqrt{a\nu_f}\,(\mathrm{x+y})^{\frac{n-1}{2}}\left[\frac{n+1}{2}\left(f + g\right) + \frac{n-1}{2}\eta\left(f' + g'\right)\right].
\end{aligned}\right\} \tag{7}$$

Equation of continuity satisfied. Eq(2-6) are converted as

$$f''' + \left(1 + K_1\right)\left(\frac{n+1}{2}\left(f + g\right)f'' - n\left[\left(f'\right)^2 + fg'\right]\right)$$
$$+K_2\left(\frac{3n-1}{2}\left(f'' + g''\right)f'' - \left(n-1\right)\left(g' + f'\right)f'''\right. \tag{8}$$
$$\left.-\left(\frac{n+1}{2}\right)\left(f + g\right)f^{(iv)}\right) - \left(1 + K_1\right)\left(M + K\right)f' - \lambda\left(\theta - Nr\phi - R_b\chi\right) = 0$$

$$\left(\left(1+R_d\left(1+(E-1)\theta\right)^3\right)\theta'\right)' + \Pr\left[Nb\theta'\phi' + Nt\theta'^2 + \frac{n+1}{2}(f+g)\theta'\right] \\ -Q_E \exp(-n\xi) = 0 \tag{9}$$

$$\phi'' + \frac{Nt}{Nb}\theta'' + Sc\left[\left(\frac{n+1}{2}\right)(f+g)\phi'\right] = 0 \tag{10}$$

$$\chi'' + Lbf\chi' - Pe\left(\phi''(\chi+\Omega) + \chi'\phi'\right) = 0 \tag{11}$$

Associated boundary conditions are converted as

$$\begin{aligned} &f=0,\ \chi=1,\ g=0,\ f'=1,\ \theta=1,\ g'=L,\ Nb\phi'+Nt\theta'=0,\ at\ \eta=0,\\ &f'\to 0, \chi\to 0,\ f''\to 0,\ g'\to 0,\ g''\to 0,\ \phi\to 0,\ \theta\to 0,\ as\ n=\infty. \end{aligned} \tag{12}$$

NUMERICAL PROCEDURE

Nonlinear differential systems (8, 9, 10, 11) with boundary constraints (12) are addressed in this section using the MATLAB bvp4c tool. Next, the higher-order model is simplified to a first-order initial value problem by adding new variables.

Let

$$\left.\begin{aligned} &f=r_1,\ f''=r_3,\ f'=r_2,\ f'''=r_4,\ f^{iv}=r_4',\ g=r_5,\ g''=r_7,\\ &g'=r_6, g'''=r_8, g^{iv}=r_8', \theta=r_9, \theta'=r_{10}, \theta''=r_{10}', \phi=r_{11},\\ &\phi'=r_{12}, \phi''=r_{12}', \chi=r_{13},\ \chi'=r_{14},\ \chi''=r_{14}'. \end{aligned}\right\} \tag{13}$$

It follows from (13), Eq. (8) becomes

$$\begin{aligned} r_4' = \frac{1}{K_2(r_1+r_5)}\left(\frac{2}{n+1}\right)\Bigg[& r_4 + (1+K_1)\left(\frac{n+1}{2}(r_1+r_5)r_3 - n\left[(r_2)^2 + r_2r_6\right]\right)\\ &-(1+K_1)(M+K)r_2 + K_2\left(\frac{3n-1}{2}(r_3+r_7)r_3 - (n-1)(r_6+r_2)r_4\right)\Bigg]\\ &-\lambda\left(r_9 - Nrr_{11} - R_br_{13}\right) \end{aligned} \tag{14}$$

$$\begin{aligned} r_{10}' = &\frac{1}{\left(1+R_d\left(1+(E-1)r_9\right)^3\right)'}\\ &\left[-\Pr\left[Nbr_{10}r_{12} + Ntr_{10}{}^2 + \frac{n+1}{2}(r_1+r_5)r_{10}\right] + Q_E\exp(-n\xi)\right] \end{aligned} \tag{15}$$

$$r_{12}' = -\frac{Nt}{Nb}r_{10}' - Sc\left[\left(\frac{n+1}{2}\right)(r_1+r_5)r_{12}\right] \tag{16}$$

$$r_{14}{}' = -Lbr_1 r_{14} + Pe\left(r_{10}{}'\left(r_{13} + \Omega\right) + r_{14} r_{12}\right) \tag{17}$$

With the accompanying boundary conditions, (12) becomes

$$\begin{aligned} &r_1 = 0, r_{13} = 1,\ r_2 = 1, r_5 = 0, r_9 = 1,\ r_6 = L,\ Nbr_{12} + Ntr_{10} = 0\ \ at\ \eta = 0, \\ &r_2 \to 0,\ r_{13} \to 0,\ r_6 \to 0, r_3 \to 0,\ r_7 \to 0,\ r_{11} \to 0,\ r_9 \to 0\ as\ n = \infty. \end{aligned} \tag{18}$$

4. FIGURES

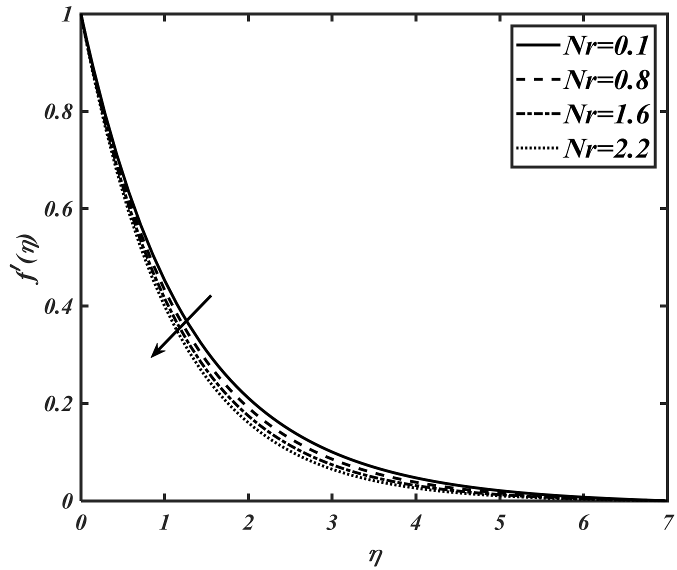

Figure 1: Impact of Nr on f'

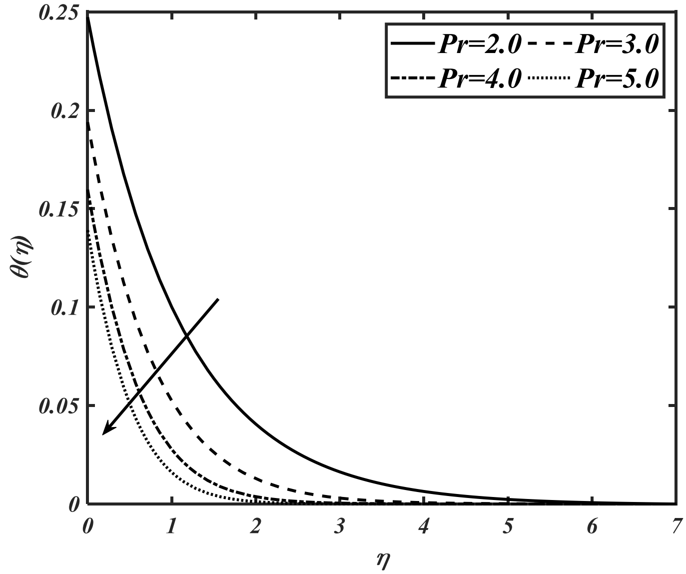

Figure 2: Impact of Pr on θ

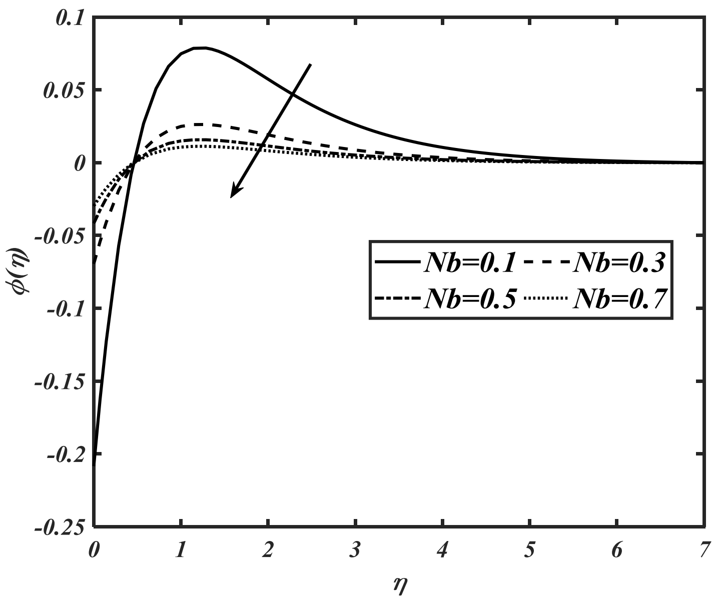

Figure 3: Impact of *Nb* on ϕ

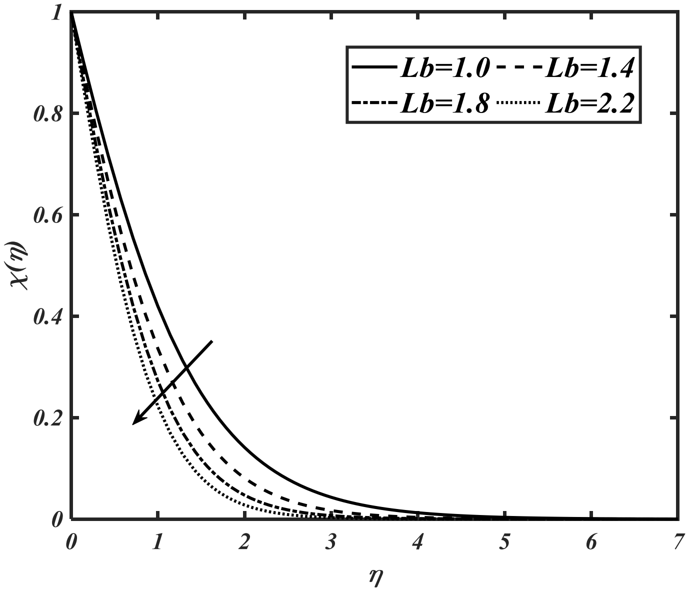

Figure 4: Impact of Lb on χ

5. RESULTS AND DISCUSSION

Figure 1 exhibits the impact of buoyancy ratio constant on the velocity of nanofluids. Increasing the value of the buoyancy ratio constant diminishes the velocity profile.

Figure 2 represents the effects of the Prandtl number on the temperature profile. The temperature of nanofluids decreases by increasing the value of the Prandtl number.

Figure 3 represents the impact of the Brownian motion parameter on concentration profile. The concentration of nanofluids decreases by an increase in the Brownian motion parameter. Bioconvection Lewis's number impact on motile microorganisms is shown in

Figure 4. Motile microorganism decreases for higher values of bioconvection Lewis number.

CONCLUSION

3-D boundary layer radiative stream of Jeffrey fluid along with thermophoresis and Brownian motion is investigated. The following are the key points of this analysis:

- The value of the velocity parameter drops when the magnetic and buoyancy ratio parameters are increased.
- Higher the amount of radiation parameter raises the temperature of the nanomaterials, but increasing the Prandtl number lowers the temperature profile.
- When the Buoyancy ratio parameter is increased, the concentration profile increases. Instead, increasing the thermophoresis parameter produces a reduction in nanoparticle concentration.

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ROLE OF TIMETABLE ON QUALITY OF EDUCATION AT PUBLIC HIGH SCHOOL LEVEL IN FAISALABAD CITY

Humaira Hina[§], Amina Afzal and **Aqsa Azam**
Department of Mathematics and Statistics
University of Agriculture, Faisalabad, Pakistan
[§]Email: humairahinafsd@gmail.com

ABSTRACT

School time table is the systematic execution of available educational resources in a specified time period .The time table is the key element in effective working of school system. Present study explores the role of time table on the quality of education. The target population of the study comprised of all the 105 public high schools and 21 higher secondary schools functioning in Faisalabad City .As per data available on official website of School Education Department, Government of the Punjab. The study is based on data collected from time table in charge teachers of public high schools in Faisalabad city as population was 252. The sample of 94 was selected by using online sampling calculator www.surveysystem.com with confidence level 95% and confidence interval 8. The simple random sampling technique was used for sample selection. The data were collected from the respondents through a well-developed questionnaire. The data analysis was performed using SPSS. According to this study 55% agreed that time table is the key element for effective working of schools, while 49% agreed the effectivity of time table in helping students to achieve high grades. About 54 % respondents agreed that teacher's qualification and training facilities were more important factors affecting scheduling in schools. Several other factors like sufficient number of classrooms, labs and library, school environment and infrastructure also affect the scheduling of school time table.

1. INTRODUCTION

A powerful administrative tool is the school timetable. Ideally, the goals and priorities of the school should be quantified by supplying the curriculum with an effective conceptual dimension. Strong reality may prevent the attainment of this ideal relation. Another purpose the school timetable serves is its allocative role. The teacher-time, pupil-time, and room-space tools have their use directly controlled by the timetable. The material resources of the facilities and equipment, which are mainly related to the subjects taught, are managed indirectly. Knowledge in these areas can allow school administrators to make informed decisions on the education process in schools and on resource allocation activities in order to achieve desired goals (Kelly and Johnson, 2005, Avella and Vasil'Ev, 2005, Dorigo and Blum, 2005).

The time table is set in such a way that students are able to learn various subjects without getting bored. For example, students can learn English for the next half hour after a period of Science. They are not getting tired of learning a specific topic in this way. There's even a lunchtime or break, where students can communicate and reconnect with each other. Also provides a time table for games and PT times to allow students to carry out physical activity along with mental exercises. In addition, periods are allocated for art, music, dance, and other classroom activities that allow students to follow their passion. A time table ensures that during learning hours of a given duration, each class has only one teacher. Even the time table eliminates the uncertainty during learning. Students are very specific on the topic they would research in a given time. In addition, teachers are also specific about what to give, and how much to offer. This preserves school order and discipline (Kremer *et al.,* 2013, Rivkin *et al.,* 2005,)

This study is very important to identify the role of timetable on the quality of education at public high schools in city Faisalabad. This research was led by the following objectives;

- To identify the major factors effecting Scheduling procedure in schools.
- To determine the role of timetable in academic performance of students.
- To explore the role of timetable in performing duties of teachers.
- To suggest the appropriate measures for the improvement of education quality through timetabling.

Wilke and Killer (2010); Babaei *et al.,* (2015) mentioned the purpose of designing a class schedule is to assign all courses for a week and match them with a single teacher to ensure that all requirements are met and then to improve the schedule's efficiency by reducing the penalty values to the maximum extent possible .Readiness and high level of willingness of teachers to perform assigned tasks. The scheduling process for educational institutions is about distributing resources over a set period of time, such as the available time for teacher shades available in the classroom. Because of the many dependencies that may arise and should be taken into account, this task can be complex and very time-consuming. That is why analytics that can help to save both time and money can help with this method.

Gul *et al.,* (2014); Post et al.; (2014) explained that focusing on time or student schedules will help resolve the recommendations that schools must have highly effective learning, curriculum, training, and evaluation systems . A considerable undertaking is to improve the quality of school facilities. School facilities have five primary facets: classroom, play grounds, lab and library, school environment. School environment and infrastructure helps to set objectives and adopt strategies.

2. METHODOLOGY

The present study entitles ‘’Role of timetable on quality of education at high schools’ level in Faisalabad city ‘’ was conducted in Faisalabad. As per data available on official website of school education department government of Punjab Lahore i.e. www.schools.punjab.gov.pk, the total number of public high schools is 105 and higher secondary schools is 21 which are functioning in city Faisalabad. The target population was 252 teachers of these schools who are the time table in charges .The sample size of this study was 94 which were selected through the website: www.survysystem.com with 95% confidence level and 8% confidence interval. The respondents were selected through simple random sampling technique .A questionnaire consisted on 20 items prepared by the researcher for the sample at high school level.

Questionnaire were prepared on 5-points Likert Scale representing strongly agree, agree, neutral, disagree and strongly disagree. The Computer software SPSS (Statistical Package for Social Sciences) was used to reach the conclusion. Frequencies, Percentage, Mean, Weighted score and Standard deviations were calculated from SPSS and used to interpret and discussion of the results.

3. RESULTS AND DISCUSSIONS

Table 1
Mean, Standard Deviation and Rank according to Academic Achievements of Students

Statement	Mean	SD	Weighted Score	Order Rank
Improper timetable grounds unorganized behavior of high school students	2.46	1.28	232	1
Due to periodic shift, motivational level of students towards their study is affected	2.27	1.17	214	2
Timing is considered a major tool to ensure the engagement level of students in study.	2.22	1.18	209	3
Time table provides a direction to students to achieve enhanced curricular performance.	1.98	.93	187	4

Scale: Strongly Disagree, Disagree, Neutral, Agree, Strongly Agree

In a previous similar study Finn *et al.*, (2014) revealed that the timetable can be an important instrument in developing a community that facilitates student attainment as there is a different interrelation among teaching and learning. In previous study Burke and Rudova (2007) said that timetable plays an important role in the academic schedule’s efficiency.

Table 2
Mean, Standard Deviation and Rank of the Respondents according to Role of Timetable in Performing Academic Duties of Teachers

Statement	Mean	SD	Weighted Score	Order Rank
Timetable guides the teaching staff to set their priorities in more appropriate way	2.24	1.00	211	1
It provides a comfort zone to teachers to perform their duties more effectively	2.14	1.08	202	2
Time table guides teachers to manage their resources efficiently	2.03	1.23	191	4
Timetable enables the teaching staff to coordinate and corporate with each other.	1.92	1.03	181	5

Scale: Strongly Disagree, Disagree, Neutral, Agree, Strongly Agree

In other research Ducharme and Harris (2005) investigated the role of timetable in performing teachers academic duties and the results showed that teachers should have the ability, within a fair timeline, to fulfil all their duties, and students should have ready access to their teachers.

Table 3
Mean, Standard Deviation and Rank Respondents according to Job Satisfaction of the Teachers

Statement	Mean	SD	Weighted Score	Order Rank
Teachers find more clarity in their tasks	2.57	1.14	242	1
.Proper timetable implementation enhances teachers' satisfaction level.	2.39	1.05	225	2
Teachers work freely and show high level of performance	2.38	1.27	224	3
Teacher's behavior towards their duties is positively influenced.	2.04	1.01	192	4

Scale: Strongly Disagree, Disagree, Neutral, Agree, Strongly Agree

As compare to other study Burke and Trick (2005); Cole *et al.*, (2000) suggested that Staff training is required for all types of timetables. Teachers, for example, need to be able to teach a range of learning styles, teach higher-order cognitive skills, use problem-solving plans, improve management skills of teachers, and use in-class technology.

Table 4
Mean, Standard Deviation and Rank of Respondents according to Identify the Facilities Effecting Scheduling Procedure in School

Statement	Mean	SD	Weighted Score	Order Rank
Availability of playground to provide recreational opportunities to students	2.34	1.31	220	1
Sufficient no of class rooms to conduct regular academic programs	2.10	.90	198	2
Availability of lab and library for additional academic activities of students	2.02	1.00	190	3

Scale: Strongly Disagree, Disagree, Neutral, Agree, Strongly Agree

As compare to another research Gul *et al.,* (2014) explained that Framework specifying that School environment and infrastructure helps to set objectives and adopt strategies. A considerable undertaking is to improve the quality of school facilities. School facilities have five primary facets: classroom, play grounds, lab and library, school environment.

Table 5
Mean, Standard Deviation and Rank of Respondent according to Scheduling Procedure

Statement	Mean	SD	Weighted Score	Order Rank
Overall performance and reputation of school is improved due to effective scheduling	2.90	1.36	273	1
Highly qualified teachers help the school management to design and implement timetable more efficiently	2.85	1.31	268	2
Scheduling procedure helps to avoid the conflicts of class replacement	2.84	1.24	267	3
Efficient utilization of available resources is possible	1.81	.87	171	4

Scale: Strongly Disagree, Disagree, Neutral, Agree, Strongly Agree

Similarly in another research Daskalakis and Birbes (2005) explained that a successful schedule clearly shows what kinds of curricular and co-curricular activities are being done at various locations and hours in the school. It is possible to contain all this information in one timetable. Overall performance and reputation of school is improved due to effective scheduling.

Timetable is really a mirror that replicates the total resources and informative program of the school. The results of this study will relate to the awareness of the efficacy of various schedules, offering knowledge that can assist teachers in evaluating choices available to facilitate academic achievements for students, which can lead to increased academic achievement and quality of education (Patall et al., 2010).

4. CONCLUSION

- The development of a class timetable for an organization and the policies it uses for the willingness of its students to obtain the developments they need to progress towards their degree goals plays a critical role in transfer students to these classes.
- Attempts to create a course timetable that enhances the ability for students to access the courses they need will create efficiencies for students as well as more effective school or graduate operations.
- For students with special needs, successful timetable would also include a variety of times.
- It is a tool that determines the successful functioning of a person or a group of individuals or an institution. The Time Table is a method of preparation that is an essential element of an institution's education or any other method of institution. In our lives, time is very important and plays a significant role.
- You will no longer suffer from stress by managing well time and your works / tasks will be performed on time and with great quality. It explains which subject the teachers will take. It is easy and saves resources and time. In order to produce a time table, most time table system uses complex design, but also compromises clarity as a result.

5. RECOMMENDATIONS

- Teachers should be given time to improve their professional development concerning the development and manipulation of timetables in schools.
- Focus on advancing academic work in subject areas, learning opportunities and education services for students through timetables.
- Include a significant amount of high quality, structured time beyond the traditional school timetable.

- As a part of creative school schedules, interactive learning is used as an opportunity to involve students in learning experiences.
- High schools must be equipped with maximum facilities to administer curricular and co-curricular activities within the given time periods.

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AL-ZUGHAIR TRANSFORM ON LINEAR DIFFERENTIAL EQUATIONS OF MOMENT PARETO DISTRIBUTION

Ali Moazzam and **Muhammad Zafar Iqbal**
Department of Mathematics and Statistics,
University of Agriculture, Faisalabad, Pakistan
Email: alimoazzam7309723@gmail.com
mzts2004@hotmail.com

ABSTRACT

Integral conversion is important in mathematical computations since it simplifies a difficult problem. It's simple to decouple a given function's characteristics from its parent function. Following is the assignment of integrating. In this paper, we use new type of transformation knows as Al-Zughair transformation to solve linear ordinary differential equation of first, second and third order of probability density function, hazard and survival function of moment Pareto distribution by using the properties of this integral transform which provides the vital approach to solve the linear ordinary differential equation.

1. INTRODUCTION

For nearly two centuries, integral transforms have been successfully used to a range of issues in arithmetic, mathematical physics, and other disciplines of study. Integral transforms, as well as the Laplace and Fourier transforms, are said to have originated in P.S. Laplace's (1749-1827) fundamental work on applied mathematics (Arnold, 2008). Here's we are going to explain the expansion of Al-Zughair transformation which is a very efficient tool to solve the ordinary differential equation and also a good approach to explain the properties of beta distribution. Al-Zughair is an integral transformation and defined as

$$Z\{f(w)\} = \int_1^e \frac{(lnw)^\omega}{w} f(w)dw = F(\omega), 1 < w < e \tag{1}$$

This integral is convergent in whole give interval and θ is a constant that is greater than (-1). Transformations play an important role in many fields of sciences (Kuffi et al., 2020). In economics, the Pareto distribution is a normally used distribution. It was first used by Pareto to explain the distribution of income among individuals. It's the appropriate model for conditions where the 80–20 rule applies, i.e. when 80% of the effect creates from 20% of the causes. Certainly, a small fraction of society's wealth is used or owned by a limited number of people. PDF (Probability Density Function) of moment Pareto distribution is

$$f(x) = (\beta - 1)\alpha^{(\beta-1)}x^{-\beta}, \alpha > 0, \alpha < x < \infty, \beta > 0. \tag{2}$$

where $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are parameters and it is represented as $\boldsymbol{x}\sim$ Pareto $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ + (Bhatti et al., 2018). The Laplace transformation is one of the most essential transformations for solving the L.O.D.E. with constant coefficients and certain beginning conditions and also be applicable in various (PDF) of different distributions (Bellman and Roth, 1984). Here's a new transformation is applied to solve L.O.D.E of Pareto distribution in the form of Euler's equation.

2. PRELIMINARIES

Definition:

Al-Zughair is an integral transformation and defined as:

$$Z\{f(w)\} = \int_1^e \frac{(lnw)^\omega}{w} f(w)dw = F(\omega), 1 < w < e$$

This integral is convergent in whole give interval and ω is a constant that is greater than (-1) (Mohammed, et al., 2016).

Lemma:

$Z\{\alpha f(w) \pm \beta g(w)\} = \alpha Z\{f(w)\} \pm \beta Z\{g(w)\}$**,** where α and β are constants and the functions $f(w)$ and $g(w)$are defined when $1 < w < e$, (Mohammed and Majde, 2019).

Theorem:

If $Z\{f(w)\} = F(w)$ and α is a constant, then,

$$Z\{(lnw)^\alpha f(w)\} = F(\omega + \alpha). \tag{3}$$

Inverse of Al-Zughair Transformation:

Let $f(w)$ be a function and w lie in interval $[1, e]$ and $Z\{f(w)\} = F(w)$ $f(w)$ is called inverse of $Z\{f(w)\}$ and will be described as

$$Z^{-1}\{F(\omega)\} = f(w). \tag{4}$$

Theorem:

$If\, Z^{-1}\{f(w)\} = F(w)$ then

$$Z^{-1}\{F(\omega + a)\} = (lnw)^\alpha Z^{-1}\{F(w)\} \tag{5}$$

where a behaves as a constant (Jaabar and Hussain 2021).

Ali's Equation:

AL-Zughair transformation is very helpful to solving a new type of linearly ordinary differential equation (L.O.D.E) with logrithmic coefficient. L.O.D.E is transform by Al-zughair transform in $F(\omega)$ an algebric expression which is further simplified by using inverse of Al-Zughair transformation and the result after simplification will be the solution of ordinary differential equation. The equation is given as

$$\begin{aligned} y = f(lnw) = a_0(lnw)^n y^{(n)} + a_1(lnw)^{n-1} y^{(n-1)} \\ + \cdots + a_{n-1}(lnw)y + a_n y \end{aligned} \tag{6}$$

where $a_0, a_1, a_2, \ldots, a_n$ are constants, and the above equation is known as Ali's Equation.

Theorem

If a function $\boldsymbol{f(lnw)}$ is defined at $\boldsymbol{w} \in [\boldsymbol{1}, \boldsymbol{e}]$ and $\boldsymbol{y^{(1)}(lny)}, \boldsymbol{y^{(2)}(lnw)}$, $\boldsymbol{y^{(3)}(lnw)}, \dots, \boldsymbol{y^{(n)}(lnw)}$ be the derivatives of the function $\boldsymbol{f(lnw)}$. Then

$$\boldsymbol{Z\{(lnw)^n y^n(lnw)\} = y^{(n-1)}(1) + (-1)^n(\omega + n)y^{(n-2)}(1)} \\ \boldsymbol{+(-1)^{n-1}\big(\omega + (n-1)\big)y^{(n-3)}(1) + \cdots + (-1)^n(\omega + n)!\,Z\{y(lnw)\}}. \quad (7)$$

3. RESULTS

Al-Zughair Transformation on First Order O.D.E Moment Pareto Distribution:

$$xf'(x) + \beta f(x) = 0 \text{ and } f(1) = (\beta - 1)\alpha^{\beta-1}, \\ f'(1) = -\beta(\beta - 1)\alpha^{\beta-1},\ f''(1) = \beta(\beta + 1)(\beta - 1)\alpha^{\beta-1} \quad (8)$$

Sol:

Substitute $x = lnw$, so equation (8) will becomes,

$$(lnw)f'(lnw) + \beta f(lnw) = 0 \quad (9)$$

by applying Al-Zughair Transform, Equation (9) will become,

$$f(1) - (\omega + 1)F(\omega) + \beta F(\omega) = 0$$

$$F(\omega) = \frac{f(1)}{(\omega + 1 - B)} \quad (10)$$

by applying inverse of transform on (10), we will get $f(lnw) = (\beta - 1)\alpha^{\beta-1}(lnw)^{-\beta}$ undo sub stitution

$$f(x) = (\beta - 1)\alpha^{(\beta-1)}(x)^{-\beta}. \quad (11)$$

Al-Zughair Transformation on Second Order O.D.E Moment Pareto Distribution:

$$f''(x) + (\beta + 1)f'^{(x)} = 0$$

and

$$f(1) = (\beta - 1)\alpha^{\beta-1},\ f'(1) = -\beta(\beta - 1)\alpha^{\beta-1}, \\ f''(1) = \beta(\beta + 1)(\beta - 1)\alpha^{\beta-1} \quad (12)$$

Sol:

Substitute $x = lnw$**,** so equation (12) will become

$$(lnw)f''(lnw) + (\beta + 1)f'(lnw) = 0$$

$$(lnw)^2 f''(lnw) + (\beta + 1)f'(lnw) = 0 \quad (13)$$

by applying Al-Zughair Transform, Equation (13) will become,

$$f'(1) - (\omega + 2)f(1) + (\omega + 2)(\omega + 1)F(\omega) + (\beta + 1)[f(1) - (\omega + 1)F(\omega) \\ = 0 - \beta(\beta - 1)\alpha^{\beta-1} - (\omega + 2)\big[(\beta - 1)\alpha^{\beta-1}\big] \\ + (\omega + 2)(\omega + 1)F(\omega) + (\beta + 1)(\beta - 1)\alpha^{\beta-1} \\ - (\beta + 1)(\omega + 1)F(\omega)] = 0$$

$$F(\omega)=\frac{\beta(\beta-1)\alpha^{\beta-1}}{(\omega+1)}+\frac{(\omega+2)(\beta-1)\alpha^{\beta-1}}{(\omega+1)}+\frac{(\beta+1)(\beta-1)\alpha^{\beta-1}}{(\omega+1)} \tag{14}$$

Applying inverse of transform of (14), we will get $f(lnw)=(\beta-1)\alpha^{\beta-1}(lnw)^{-\beta}=0$ undo substitution

$$f(x)=(\beta-1)\alpha^{\beta-1}(x)^{-\beta} \tag{15}$$

Al-Zughair Transformation on Third Order O.D.E Moment Pareto Distribution:

$$xf'''(x)+(\beta+2)f''(x)=0$$

and

$$f(1)=(\beta-1)\alpha^{\beta-1},\ f'(1)=-\beta(\beta-1)\alpha^{\beta-1},\ f''(1)=\beta(\beta+1)(\beta-1)\alpha^{\beta-1} \tag{16}$$

Sol:

Substitute $x=lnw$**,** so equation (16) will become,

$$(lnw)f'''(lnw)+(\beta+2)(lnw)f'(lnw)=0$$

$$(lnw)^3f'''(lnw)+(\beta+2)lnw^2f''(lnw)=0 \tag{17}$$

by applying Al-Zughair Transform, Equation (17) will become,

$$f''(1)-(\omega+3)f'(1)+(\omega+2)(\omega+1)f(1)-(\omega+3)(\beta+2)(\omega+1)F(\omega)$$
$$+(\beta+2)[f'(1)-(\omega+2)f(1)+(\omega+2)(\omega+1)F(\omega)]=0$$

$$(\beta-1)\alpha^{\beta-1}[\beta(\beta-1)+(\omega+3)\beta+(\omega+3)(\omega+2)-\beta(\beta+2)$$
$$-(\beta+2)(\omega+2)]$$

$$=(\omega+3)(\omega+2)(\omega+(1-\beta))F(\omega)$$

$$F(\omega)=\frac{(\beta-1)\alpha^{\beta-1}(\omega^2+3\omega+2)}{(\omega^2+3\omega+2)(\omega+(1-\beta))} \tag{18}$$

by applying inverse of transform of (18), we will get $mf(lnw)=(\beta-1)\alpha^{\beta-1}lnw^{-\beta}$, undo substitution

$$f(x)=(\beta-1)\alpha^{\beta-1}(x)^{-\beta} \tag{19}$$

Al-Zughair Transformation on First Order O.D.E of Survival Function of Moment Pareto Distribution:

$$xf'(x)+(\beta+1)f(x)=0\ and\ f(1)=\alpha^{(\beta-1)}, f'(1)=(\beta+1)\alpha^{\beta-1},$$

$$f''(1)=(\beta+1)(\beta+2)\alpha^{\beta-1}, \tag{20}$$

Sol: Substitute $x=lnw$**,** so equation (20) will become

$$(lnw)f'(lnw)+(\beta+1)f(lnw)=0 \tag{21}$$

by applying Al-Zughair Transform, Equation (21) will become,

$$f(1)-(\omega+1)F(\omega)+(\beta+1)F(\omega)=0$$

$$[\omega+\{1-(\beta+1)\}]F(\omega)=f(1)$$

$$F(\omega)=\frac{f(1)}{(\omega+(1-(1+B))} \quad (22)$$

by applying inverse of transform of (22), we will get $f(lnw)=\alpha^{\beta-1}(lnw)^{-(\beta+1)}$, undo substitution

$$f(x)=\alpha^{(\beta-1)}(x)^{-(\beta+1)}. \quad (23)$$

Al-Zughair Transformation on Second Order O.D.E of Survival Function of Moment Pareto Distribution:

$$xf''(x)+(\beta+1)f'(x)=0$$

and

$$f(1)=\alpha^{\beta-1},\ f'(1)=-(\beta+1)\alpha^{\beta-1},\ f''(1)=(\beta+1)(\beta+2)\alpha^{\beta-1} \quad (24)$$

Sol:

Substitute $x=lnw$**,** so equation (24) will become

$$(lnw)f''(lnw)+(\beta+2)f'(lnw)=0$$

$$(lnw)^2f''(lnw)+(\beta+2)(lnw)f'(lnw)=0 \quad (25)$$

by applying Al-Zughair Transform, Equation (25) will become,

$$\begin{aligned} f'(1)-(\omega+2)f(1)+(\omega+2)(\omega+1)F(\omega)+(\beta+2)[f(1)-(\omega+1)F(\omega) \\ =0-(\beta-1)\alpha^{\beta-1}-(\omega+2)\alpha^{\beta-1}+(\omega+2)(\omega+1)F(\omega) \\ +(\beta+2)\alpha^{\beta-1}-(\beta+2)(\omega+1)F(\omega)=0 \end{aligned}$$

$$F(\omega)=\alpha^{\beta-1}\frac{(\omega+1)}{(\omega+1)(\omega+(1-(\beta+1)))} \quad (26)$$

by applying inverse of transform of (26), we will get $f(lnw)=\alpha^{\beta-1}(lnw)^{-(\beta+1)}$, undo substitution

$$f(x)=\alpha^{\beta-1}(x)^{-(\beta+1)}, \quad (27)$$

Al-Zughair Transformation on Third Order O.D.E of Survival Function of Moment Pareto Distribution:

$$xf'''(x)+(\beta+3)f'(x)=0$$

and

$$f(1)=\alpha^{\beta-1},\ f'(1)=-(\beta-1)\alpha^{\beta-1},\ f''(1)=(\beta+1)(\beta+2)\alpha^{\beta-1} \quad (28)$$

Sol:

Substitute $x=lnw$ so equation (28) will become,

$$(lnw)f'''(lnw)+(\beta+3)(lnw)f''(lnw)=0$$

$$(lnw)^3f'''(lnw)+(\beta+3)(lnw)^2f''(lnw)=0 \quad (29)$$

by applying Al-Zughair Transform, Equation (29) will become,

$$f''(1)-(\omega+3)f'(1)+(\omega+2)(\omega+1)f(1)-(\omega+3)(\beta+2)(\omega+1)F(\omega)+(\beta+3)$$

$$\alpha^{\beta-1}[(\beta+2)(\beta+1)+(\omega+3)(\beta+1)+(\omega+3)(\omega+2)-(\beta+1)(\beta+3)-(\beta+3)(\omega+2)]=(\omega+3)(\omega+2)(\omega+(1-(\beta+1)))F(\omega)$$

$$F(\omega)=\frac{\alpha^{\beta-1}(\omega^2+3\omega+2)}{(\omega^2+3\omega+2)(\omega+(1-(\beta+1)))} \quad (30)$$

by applying inverse of transform of (30), we will get $f(lnw)=\alpha^{\beta-1}lnw^{-(\beta+1)}$, undo substitution

$$f(x)=\alpha^{\beta-1}(x)^{-(\beta+1)} \quad (31)$$

Al-Zughair Transformation on First Order O.D.E of Hazard Function of Moment Pareto Distribution:

$$xf'(x)+f(x)=0 \; and \; f(1)=\beta-1=f''(1)=-(\beta-1) \quad (32)$$

Sol:

Substitute $x=lnw$, so equation (32) will become,

$$(lnw)f'(lnw)+f(lnw)=0 \quad (33)$$

by applying Al-Zughair Transform, Equation (33) will become,

$$f(1)-(\omega+1)F(\omega)+F(\omega)=0$$

$$F(\omega)=\frac{f(1)}{(\omega+(1-1))} \quad (34)$$

by applying inverse of transform of (34), we will get $f(lnw)=\frac{\beta-1}{lnw}$, undo substitution

$$f(x)=\frac{\beta-1}{x} \quad (35)$$

Al-Zughair Transformation on Second Order O.D.E of Hazard Function of Moment Pareto Distribution:

$$xf''(x)+2f'(x)=0 \; and \; f(1)=\beta-1=f'(1)=-(\beta-1) \quad (36)$$

Sol:

Substitute $x=lnw$**,** so equation (36) will become,

$$(lnw)f''(lnw)+2f'(lnw)=0$$

$$(lnw)^2f''(lnw)+2(lnw)f'(lnw)=0 \quad (37)$$

by applying Al-Zughair Transform, Equation (37) will become,

$$f'(1)-(\omega+2)f(1)+(\omega+2)(\omega+1)F(\omega)+2(\omega+1)f(1)-2(\omega+1)F(\omega)=0$$

$$F(\omega) = \frac{(\beta-1)}{\omega+(1-1)} \tag{38}$$

by applying inverse of transform of (38), we will get $f(lnw = \frac{(\beta-1)}{(lnw)}$, undo substitution

$$f(x) = \frac{(\beta-1)}{x} \tag{39}$$

Al-Zughair Transformation on Third Order O.D.E of Hazard Function of Moment Pareto Distribution:

$$xf'''(x) + 3f'(x) = 0 \text{ and } f(1) = \beta - 1 = f''(1),\ f'(1) = -(\beta - 1) \tag{40}$$

Sol:

Substitute $x = lnw$**,** so equation (40) will become

$$(lnw)f'''(lnw) + 3f''(lnw) = 0$$

$$(lnw)^3 f'''(lnw) + 3(lnw)^2 f''(lnw) = 0 \tag{41}$$

by applying Al-Zughair Transform,

$$f''(1) - (\omega + 3)f'(1) + (\omega + 2)(\omega + 1)f(1) - (\omega + 2)(\omega + 1)(\omega + 3)F(\omega) + 3f'(1) - 3(\omega + 2)f(1) + (\omega + 2)(\omega + 1)F(\omega) = 0$$

$$F(\omega) = \frac{(\beta-1)}{\omega+(1-1)} \tag{42}$$

by applying inverse of transform of (42), we will get $f(lnw = \frac{(\beta-1)}{(lnw)}$ undo substitution

$$f(x) = \frac{(\beta-1)}{x}. \tag{43}$$

CONCLUSION

A new type of transformation knows as Al-Zughair transformation that is use to solve partial differential equations and linear differential equations of different orders. In this paper we solve the ordinary differential equation of first order, second order and third order of probability density function which gives the accurate results. We also find the solution of ordinary linear differential equations of hazard function and survival function of moment Pareto distribution by utilizing the properties of Al- Zughair transformation which also gives the correct results and satisfied the properties of this known transformation.

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ON CONFORMABLE FRACTIONAL CONFLUENT HYPERGEOMETRIC FUNCTIONS

Muhammad Zafar Iqbal, Asad Ali, Afeen Azam and **Adeel Anjum**
Department of Mathematics and Statistics
University of Agriculture, Faisalabad, Pakistan
Email: mzts2004@hotmail.com
mrasadali5100@gmail.com
afeenm95@gmail.com
adeelanjum1954@gmail.com

ABSTRACT

In this paper, we define fractional confluent hypergeometric and solve its differential equation by using the fractional power series technique. Also, we introduce the fractional Kummer's formula and discuss its conformable fractional derivative and its integral representation.

1. INTRODUCTION

Fractional calculus is classified as generalized fractional integrals or derivatives. In last few decades, various researchers and mathematicians have paid their attention towards the fractional calculus and its applications in different fields of science. Because, fractional order models are more realistic than the classical models of the integer order. Fractional derivatives are an excellent tool for describing memory and the genetic properties of different materials. The advantages of fractional derivatives are eliminate the deficiency emerging from the occurrence of continuum flow of traffic in the modeling of fluid dynamics. Fractions integrals and derivatives also appear in the theory of control of dynamic systems when a controlled system or controller is described by differential equations of the fraction (Hammad et al., 2019; Podlubny, 1999). The Grunwald Letnikov, the Riemann-Liouvillë and the Caputo fractional derivatives, and also called the sequential fractional derivative were the main tools to achieve the results. Several major works on various aspects of fractional accounts, in which the original definitions of fractional differentiations are use for formulating and solving problems, we may refer to (Hammad et al., 2019; Abdeljawad, 2015; Hammad et al., 2019; Kummer, 1836). In Khalil et al. (2014) introduced a new concept of a simple fractional derivative, named as the conformable fractional derivative. This new definition is an elegant extension of the classical derivative and seems to be the most convenient one in fractional calculus. This new definition has been developed by Khalil et al. (2014) and recently, it has been modified. For more developments on the conformable differentiation, we refer to (Karlsson, 1971; Ali et al., 2020; Rainville, 1960).

Definition 1.1

In Khalil et al. (2014) For a function $f:(m,\infty)- \to R$, the left conformable fractional derivative of f order α is defined as

$$(D_m^\alpha f)(\zeta) = \lim_{\varepsilon \to 0} \frac{f(\zeta + \varepsilon(\zeta - a)^{1-\alpha}) - f(\zeta)}{\in}, \tag{1.1}$$

for all $\zeta > m$, $\alpha \in (0,1]$. *When* $m = 0$, *it can be written as* D^α. *If* $(D^\alpha f)(\zeta)$ *exists on* (m,n), *then* $(D^\alpha f)(m) = \lim (D^\alpha f)(\zeta)$.

$\zeta \!-\!\!\longrightarrow m^+$.

Definition 1.2

In Khalil et al. (2014) For a function $f:(-\infty,n) -\!\!\to R$, *the right conformable fractional derivative of f order* α *is defined as*

$$(D_n^\alpha f)(x) = \lim_{\varepsilon \to 0} \frac{f(\zeta + \varepsilon(b - \zeta)^{1-\alpha}) - f(\zeta)}{\in},, \tag{1.2}$$

for all $\zeta < n$, $\alpha \in (0,1]$. *If* $(D^\alpha f)(\zeta)$ *exists on* (m,n), *then* $(D^\alpha f)(n) = (D^\alpha f)(m) = \lim (D^\alpha f)(\zeta)$.

$\zeta \!-\!\!\longrightarrow n^{--}$.

Theorem 1.1

In Khalil et al. (2014) Let $\alpha \in (0,1]$ *and f,g be* α*−differentiable at a point* $x > 0$. *Also let* $D = \frac{d}{dx}$. *Then*

i) $D^\alpha(cf + dg) = cD^\alpha f + dD^\alpha g$;

ii) $D^\alpha(fg) = gD^\alpha f + fD^\alpha g$;

iii) $D^\alpha(\frac{f}{g}) = (gD^\alpha f - fD^\alpha g)g^{-2}$;

iv) $D^\alpha(1) = 0$;

v) $D^\alpha f(\varphi) = (\varphi - m)^{1-m} f'(\varphi)$ *if in addition f is differentiable.*

Theorem 1.2

Khalil et al. (2014) Let f is an infinitely α*−differentiable function for some* $\alpha \in (0,1]$ *at a neighborhood of a point* ζ_o. *Then f has the fractional power series expansion:*

$$f(\zeta) = \sum_{k=0}^{\infty} \frac{{}^{(k)}D_{\zeta_o}^\alpha f)(\zeta_o)(\zeta - \zeta_o)^{k\alpha}}{\alpha^k k!}, \zeta_o < \zeta < \zeta_o + \mathrm{R}^{\frac{1}{\alpha}}, \mathrm{R} > 0. \tag{1.3}$$

Here, ${}^{(k)}D_{\zeta_o}^\alpha f)(\zeta_o)$ *means the fractional derivative k times.*

2. PRELIMINARIES

The confluent hypergeometric function ${}_1F_1(\eta;\xi;\omega)$ satisfies the following differential equation (Ali et al., 2020; Rainville, 1960; Shani et al., 2016)

$$\omega\frac{d^2X}{d\omega^2}+(\xi-\eta)\frac{dX}{d\omega}-\eta X=0. \tag{2.4}$$

The regular singular point for the equation (2.4) is $\omega = 0$. If we put $\xi = 2\eta$, change the variable $\omega \to 2z$ and let $\chi = e^{zy}$, then (2.4) becomes

$$z\frac{d^2y}{dz^2}+2\eta\frac{d^2y}{dz^2}-yz=0. \tag{2.5}$$

one of the solution is $(when, 2\eta \neq\ 0,-1,-2,...)$

$$y=e^{-z}{}_1F_1\left(\eta;2\eta;2z\right). \tag{2.6}$$

The equation (2.5) is invariant, when we replace z to $-z$. Hence, the new independent variable $\sigma=\frac{z^2}{4}$,

$$\sigma\frac{d^2y}{d\sigma^2}+(a+\frac{1}{2})\sigma\frac{d^2\sigma}{dz^2}-\sigma z=0.\ , \tag{2.7}$$

(2.5) is differential equation for the ${}_0F_1$ function with denominator parameter $(\eta+\frac{1}{2})$ and argument $\sigma=\frac{z^2}{4}$. Hence, if $(\eta+\frac{1}{2})$ is non-integral, the general solution of (2.7) is

$$y=M_o F_1(-;\eta+\frac{1}{2};\frac{z^2}{4})+N(z^2)^{\frac{1}{2}-\eta}{}_0F_1(-;\frac{3}{2}-\eta\frac{z^2}{4}), \tag{2.8}$$

But, equation (2.7) also has the solution (2.6). Hence,

$$e^{-z}{}_1F_1\left(\eta;2\eta;2z\right)=M_o F_1(-;\eta+\frac{1}{2};\frac{z^2}{4})+N(z)^{1-2\eta}{}_0F_1(-;\frac{3}{2}-\eta;\frac{z^2}{4})\ . \tag{2.9}$$

In (2.9), the first member on the right-hand side and left-hand side are both analytic at $z = 0$, but the remaining term is not, due to the presence of the factor $z^{1-2\eta}$. When 2η is an odd positive integer, the second solution in (2.7) involves a *logz* term, and the same argument shows that $M = 1$ and $N = 0$, as in (2.9).

3. SOLUTIONS OF FRACTIONAL CONFLUENT HYPERGEOMETRIC DIFFERENTIAL EQUATION

In this section, we arise the fractional confluent hypergeometric differential equation by supposing ${}_1F_1\left(\eta;\xi;\omega^\alpha\right)$ as a one solution of the desired equation. The operator $\theta=\omega^\alpha\frac{d}{d\omega^\alpha}$ is helpful to derive a differential equation of fractional confluent hypergeometric

$$y = {}_1F_1\left(\eta;\xi;\omega^{\alpha}\right) = \sum_{n=0}^{\infty} \frac{(\eta)_n (\omega)^{\alpha n}}{(\xi)_n n!}. \tag{3.10}$$

From (3.10), we find that

$$\theta(\theta+\alpha(\xi-1))y = \theta \sum_{n=0}^{\infty} \frac{(\eta)_n}{(\xi)_n} \frac{(\alpha n+\alpha(\xi-1))\omega^{\alpha n}}{n!}$$

$$= \alpha \sum_{n=1}^{\infty} \frac{(\eta)_n}{(\xi)_{n-1}} \frac{\alpha\omega^{\alpha n}}{(n-1)!}.$$

By shifting index, we obtain that

$$\theta(\theta+\alpha(\xi-1))\omega = \alpha^2\omega^{\alpha} \sum_{n=0}^{\infty} \frac{(\eta)_{n+1}}{(\xi)_n} \frac{\omega^{\alpha n}}{n!}$$

$$= \alpha\omega^{\alpha}(\theta+\alpha\eta) \sum_{n=0}^{\infty} \frac{(\eta)_n}{(\xi)_n} \frac{\omega^{\alpha n}}{n!}$$

This gives the following form

$$\theta\left(\theta+\alpha\left(\xi-1\right)\right)y = \alpha\omega^{\alpha}\left(\theta+\alpha\eta\right)y.$$

Thus, we have shown that $\omega = {}_1F_1\left(\eta;\xi;\omega^{\alpha}\right)$ is a solution of the differential equation

$$\theta\left(\theta+\alpha\left(\xi-1\right)\right)y - \alpha\omega\,\omega^{\alpha}\left(\theta+\alpha\eta\right)y = 0 \tag{3.11}$$

where $\theta = \omega^{\alpha}\dfrac{d}{d\omega^{\alpha}}$, and $\alpha \in (0,1]$. Since

$$\theta\left(y\right) = \omega^{\alpha}D^{\alpha}y \;\; \theta\left(\theta-1\right)y = \omega^{2\alpha}D^{\alpha}D^{\alpha}y,$$

the equation (3.11) can be rewritten in the form of

$$\omega^{\alpha}D^{\alpha}D^{\alpha}y + \alpha\left(\xi-\omega^{\alpha}\right)D^{\alpha}y - \alpha^2\eta y = 0. \tag{3.12}$$

We give the series solution of differential equation of conformable confluent hypergeometric by using the fractional power series.

Definition 3.1

In Khalil et al. (2014) the point $\omega = 0$ is an α-regular singular point for

$$\omega^{\alpha}D^{\alpha}D^{\alpha}y + P\left(x\right)D^{\alpha}y + Q\left(x\right)y = 0.$$

$$\textit{if} \lim_{\omega\to 0}\frac{P(x)}{\omega^{\alpha}} \;\; \textit{exists.}$$

Clearly, $\omega = 0$ is an α-regular singular point for the desired equation.

Definition 3.2

In Khalil et al. (2014), the fractional power series is defined as

$$\sum_{n=0}^{\infty} a_n \omega^{n\alpha}.$$

We will use $D^{n\alpha}$ to denote $D^{\alpha}...D^{\alpha}$ n-times. If $D^{n\alpha}f$ exists for all n in some interval $[0,m]$, then the function f can be written in the form of a fractional power series. Let $y = \sum_n a_n (n\alpha + m)\omega^{n\alpha+m-\alpha}$. From basic properties of the conformable fractional derivative, we have

$$D^{\alpha} y = \sum_{n=0}^{\infty} a_n (n\alpha + m)\omega^{n\alpha+m-\alpha},$$

and

$$D^{\alpha} D^{\alpha} y = \sum_{n=0}^{\infty} a_n \left(n\alpha + m\right)\left(n\alpha + m - \alpha\right)\omega^{n\alpha+m-2\alpha}.$$

Substituting the above in Equation (3.12), we get the following

$$\omega^{\alpha} \sum_{n=0}^{\infty} a_n (n\alpha + m)(n\alpha + m - \alpha)\omega^{n\alpha+m-2\alpha}$$
$$+\alpha\left(\xi - \omega^{\alpha}\right) \sum_{n=0}^{\infty} a_n (n\alpha + m)\omega^{n\alpha+m-\alpha} - \alpha^2\eta \sum_{n=0}^{\infty} a_n \omega^{n\alpha+m} = 0$$

$$\sum_{n=0}^{\infty} [(n\alpha + m)(n\alpha + m - \alpha) + \alpha\xi(n\alpha + m)] a_n \omega^{n\alpha+m-\alpha}$$
$$- \sum_{n=0}^{\infty} \left[(n\alpha + m)\alpha + \alpha^2\eta\right] a_n \omega^{n\alpha+m} = 0$$

Hence, from the first term, we have

$$m\left[m - \alpha + \alpha\xi\right] a_0 = 0,$$

which is an indicial equation. Since $a_0 \neq 0$, we have

$$m\left[m - \alpha + \alpha\xi\right] = 0.$$

Hence, the solutions of the above equation are given

$m = 0$, *and* $m = \alpha - \alpha\xi$.

Also, from the rest of the terms, we have

$$a_{n+1} = \frac{(n\alpha + m)\alpha + \alpha^2\eta}{[(n+1)\alpha + m][(n+1)\alpha + m - \alpha + \alpha\xi]} a_n. \tag{3.13}$$

Putting $m = 0$, one gets

$$a_{n+1} = \frac{(n\alpha)\alpha + \alpha^2\eta}{[(n+1)\alpha][(n+1)\alpha - \alpha + \alpha\xi]} a_n.$$

Now, simplify a_{n+1} in terms of a_0 instead of a_n. For this, let $n = 0$

$$a_1 = \frac{\eta}{(1)(\xi)} a_0.$$

For $n = 1$,

$$a_2 = \frac{\eta(\eta+1)}{(1)(\xi)2(\xi+1)} a_0.$$

For $n = 2$,

$$a_3 = \frac{\eta(\eta+1)(\eta+2)}{(1)(\xi)2(\xi+1)3(\xi+2)} a_0..$$

From a repetitive approach, we have the following

$$a_n = \sum_{n=0}^{\infty} \frac{a_n}{(b)_n n!} a_0$$

Therefore, our assumed solution is

$$Y_1 = a_0 \sum_{n=0}^{\infty} \frac{(\eta)\omega^{n\alpha}}{(\xi)_n n!} \tag{3.14}$$

Putting $m = \alpha - \alpha\xi$ in (3.13)

$$a_{n+1} = \frac{(n\alpha + \alpha - \alpha\xi)\alpha + \alpha^2\eta}{[(n+1)\alpha + \alpha - \alpha\xi][(n+1)\alpha + \alpha - \alpha\xi + \alpha + \alpha\xi]} a_n$$

$$a_{n+1} = \frac{(n+\eta+1+\xi)}{(n+2-\xi)(n+1)} a_n$$

For $n = 0$,

$$a_1 = \frac{(\eta+1+\xi)}{(2-\xi)(1)} a_0$$

For $n = 1$,

$$a_2 = \frac{(\eta+1+\xi)(\eta+2+\xi)}{(2-\xi)(1)(1-\xi)(2)} a_0$$

again, from repetitive approach, we have the following

$$a_n = \sum_{n=0}^{\infty} \frac{(\eta+1+\xi)_n}{(2-\xi)_n n!} a_0$$

Another independent solution is

$$Y_2 = a_0 \sum_{n=0}^{\infty} \frac{(\eta+1+\xi)_n}{(2-\xi)_n\, n!} \omega^{n\alpha}. \tag{3.15}$$

The general solution is:

$$y = A\sum_{n=0}^{\infty} \frac{(\eta)\omega^{n\alpha}}{(\xi)_n\, n!} + B\omega^{\alpha-\alpha\omega} \sum_{n=0}^{\infty} \frac{(\eta+1+\xi)_n}{(2-\xi)_n\, n!}\omega$$

Also,

$$s\ y = A{}_1F_1\left(\eta;\xi;\omega^{\alpha}\right) + \omega^{\alpha-\alpha\xi} B{}_1F_1\left(\eta+1-\xi;2-\xi;\omega^{\alpha}\right), \tag{3.16}$$

where A and B are arbitrary constants.

KUMMER'S FORMULA

We obtain results which are characteristic of the ${}_1F_1$. We also present conformable fractional Kummer's first and second formulas.

Theorem 3.1

If ξ is neither zero nor a negative integer and $\alpha \in (0,1]$, then

$$1F_1\left(\eta;\xi;z^{\alpha}\right) = e^{z}{}_1F_1\left(\xi-\eta;\xi;-z^{\alpha}\right). \tag{3.17}$$

Proof:

Consider the product

$$e^{-z^{\alpha}}{}_1F_1\left(\eta;\xi;z^{\alpha}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n (z)^{\alpha n}}{n!} \sum_{n=0}^{\infty} \frac{(\eta)_n z^{\alpha n}}{(\xi)_n\, n!} \tag{3.18}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^k (\eta)_k z^{\alpha(n+k)}}{(\xi)_k\, k!\, n!}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(-1)^k (\eta)_k z^{\alpha n}}{(\xi)_k\, k!(n-k)!}$$

Since $\dfrac{1}{(n-k)!} = \dfrac{(-1)^k (-n)_k}{n!}$, we may write

$$e^{-z^{\alpha}}{}_1F_1\left(\eta;\xi;z^{\alpha}\right) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(-n)_k (\eta)_k}{(\xi)_k\, k!} . \frac{(-1)^n z^{\alpha n}}{n!}$$

$$= \sum_{n=0}^{\infty} {}_2F_1\left(-n;\eta;\xi;1\right). \frac{(-1)^n z^{\alpha n}}{n!}$$

It is a well-known identity

$$ {}_2F_1\left(-n;\eta;\xi;1\right)=\frac{(\xi-\eta)_n}{(\xi)_n}\ .$$

Then

$$e^{-z^{\alpha}}{}_1F_1\left(\eta;\xi;z^{\alpha}\right)=\sum_{n=0}^{\infty}\frac{(\xi-\eta)_n}{(\xi)_n}.\frac{(-1)^n z^{\alpha n}}{n!}\ ,$$

which completes our proof.

Theorem 3.2

If a is neither zero nor a negative integer and $\alpha\in(0,1]$, *then*

$$ {}_1F_1\left(\eta;2\eta;2z^{\alpha}\right)=e^{z^{\alpha}}{}_0F_1\left(-;\eta+\frac{1}{2};\frac{z^{2\alpha}}{4}\right). \tag{3.19}$$

Proof:

Since $\dfrac{1}{(n-2k)!}=\dfrac{(-1)^{2k}(-n)_{2k}}{n!}$, we have

$$e^{z^{\alpha}}{}_0F_1\left(-;\eta+\frac{1}{2};\frac{z^{2\alpha}}{4}\right)=\sum_{n=0}^{\infty}\frac{(z)^{\alpha n}}{n!}\sum_{n=0}^{\infty}\frac{(\frac{z^{2\alpha}}{4})^{2\alpha n}}{(\eta+\frac{1}{2})_n n!} \tag{4.20}$$

$$e^{z^{\alpha}}{}_0F_1\left(-;\eta+\frac{1}{2};\frac{z^{2\alpha}}{4}\right)=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{z^{\alpha(n+2k)}}{2^{2k}(\eta+\frac{1}{2})_k k!n!}$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{n}{2}]}\frac{z^{\alpha n}}{2^{2k}(\eta+\frac{1}{2})_k k!(n-2k)!}$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{n}{2}]}\frac{(-n)_{2k}z^{\alpha n}}{2^{2k}(\eta+\frac{1}{2})_k k!(n)!}$$

$$=\sum_{n=0}^{\infty}{}_2F_1\left(-\frac{n}{2};\frac{-n+1}{2};\eta+\frac{1}{2};1\right)\frac{z^{\alpha n}}{(n)!}\ .$$

Since

$$ {}_2F_1\left(-\frac{n}{2};\frac{-n+1}{2};\eta+\frac{1}{2};1\right)=\frac{2^n(\eta)_n}{(2\eta)_n}$$

one writes

$$e^{z^{\alpha}}{}_0F_1\left(-;\eta+\frac{1}{2};\frac{z^{2\alpha}}{4}\right)=\sum_{n=0}^{\infty}\frac{(\eta)_n(2z^{\alpha})^n}{(2\eta)_n\,n!}.$$

which completes our proof.

Next, we obtain a conformable fractional derivative of Kummer's formula.

Theorem 3.3

The conformable fractional derivative of Kummer's formula is

$$\frac{d^{\alpha}}{d\omega^{\alpha}}\left[e^{-z^{\alpha}}{}_1F_1\left(\eta;\xi;z^{\alpha}\right)\right]=-\frac{\alpha(\xi-\eta)}{\xi}e^{-z^{\alpha}}{}_1F_1\left(\eta;\xi+1;z^{\alpha}\right) \tag{3.21}$$

Proof:

Since

$$e^{-z\alpha}{}_1F_1\left(\eta;\xi;z^{\alpha}\right)={}_1F_1\left(\xi-\eta;\xi;-z^{\alpha}\right),$$

we have

$$\frac{d^{\alpha}}{d\omega^{\alpha}}\left[e^{-z^{\alpha}}{}_1F_1\left(\eta;\xi;z^{\alpha}\right)\right]=\frac{d^{\alpha}}{d\omega^{\alpha}}\left[\sum_{n=0}^{\infty}\frac{(\xi-\eta)_n(-1)^n z^{\alpha n}}{(\xi)_n\,n!}\right]$$

$$=\sum_{n=1}^{\infty}\frac{(\xi-\eta)_n\,\alpha(-1)^n z^{\alpha n-\alpha}}{(\xi)_n(n-1)!}$$

$$=\sum_{n=0}^{\infty}\frac{(\xi-\eta)_{n+1}\alpha(-1)^{n+1}z^{\alpha n}}{(\xi)_{n+1}(n)!}.$$

Since $\left(\xi-\eta\right)_{n+1}=\left(\xi-\eta\right)\left(\xi-\eta+1\right)$

$$\frac{d^{\alpha}}{d\omega^{\alpha}}\left[e^{-z^{\alpha}}{}_1F_1\left(\eta;\xi;z^{\alpha}\right)\right]=-\frac{\alpha(\xi-\eta)}{\xi}\sum_{n=0}^{\infty}\frac{(\xi+1-\eta)_n\,\alpha(-1)^n z^{n\alpha}}{(\xi+1)_n(n)!}.$$

Theorem 3.4

The conformable fractional of Kummer's formula has an integral representation of the form

$$e^{-z^{\alpha}}{}_1F_1\left(\eta;\xi;z^{\alpha}\right)=\frac{\Gamma(\xi)}{\Gamma(\eta)\Gamma(\xi-\eta)}\int_0^1 e^{(t-1)\alpha}t^{\eta-1}(1-t)^{\xi-\eta-1}dt \tag{3.22}$$

Proof:

We have

$$\beta(\eta+n,\xi-\eta)=\frac{\Gamma(\eta+n)\Gamma(\xi-\eta)}{\Gamma(\xi+n)}$$

$$\frac{(\eta)_n \Gamma(\eta)\Gamma(\xi-\eta)}{(\xi)_n \Gamma(\xi)} = \int_0^1 t^{\eta+n-1}(1-t)^{\xi-\eta-1} dt$$

$$e^{-z^\alpha} \sum_{n=0}^{\infty} \frac{(\eta)_n z^{n\alpha}}{(\xi)_n n!} = \frac{\Gamma(\xi)}{\Gamma(\eta)\Gamma(\xi-\eta)} \int_0^1 e^{-z^\alpha} t^{\eta+n-1}(1-t)^{\xi-\eta-1} \sum_{n=0}^{\infty} \frac{\left(z^\alpha\right)^n}{n!} dt$$

$$= \frac{\Gamma(\xi)}{\Gamma(\eta)\Gamma(\xi-\eta)} \int_0^1 e^{-z^\alpha} t^{\eta-1}(1-t)^{\xi-\eta-1} \sum_{n=0}^{\infty} \frac{\left(tz^\alpha\right)^n}{n!} dt .$$

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BAYESIAN ANALYSIS OF GOMPERTZ LOMAX DISTRIBUTION

Muhammad Zafar Iqbal[1], **Moheez Hassan**[1], **Nasir Abbas**[2]
Adnan Athar[1] and **Ashraf Ali**[1]

[1] Department of Mathematics and Statistics
University of Agriculture, Faisalabad, Pakistan
Email: mzts2004@uaf.edu.pk
ashrafaliuaf@gmail.com
moheez.hassan.mh@gmail.com
adnanatharaxis@gmail.com

[2] Department of Statistics, Government Graduate College
Jhang, Pakistan. Email: nabbasgcj@yahoo.com

ABSTRACT

In this article, a four-parameter Gompertz-Lomax (GoLom) distribution has been used for the Bayesian analysis to determine the posterior distribution, which is new evidence. The parameters of the Go-Lom Distribution have also been estimated through the classical and Bayesian approaches. The Bayesian method is a non-classical estimation method used in the field of statistical estimation to data assessment and is very valuable in real-life complications that custom to scrutinize future predictions. Bayes estimates and their associated MSEs are obtained using informative and non-informative priors under square error loss function (SELF), Quadratic Loss Function (QLF), Weighted square error loss function (WSELF), Precautionary loss function (PLF), K Loss Function (KLF), DeGroot loss function (DLF), and Al-Bayyati's loss function (ABLF). The gamma and exponential distributions have been used as informative priors. The maximum likelihood estimates are also evaluated to learn about non-informative prior. The Bayesian estimates are compared by using mean square error (MSE) of different sample sizes through a simulation study and the results are presented graphically. The estimates are obtained using the R software.

KEYWORDS

Gompertz Lomax Distribution; Priors; Loss Function; Bayes Estimators; MLE; Simulation MSE.

1. INTRODUCTION

The GoLom distribution that is more flexible as compared to the having existence distribution and is a heavy-tail distribution has changed into the new trend in distribution theory. It is also considered valuable in engineering and survival analysis, reliability, and so on [1]. In this current study, the distribution of Lomax was extended by utilizing the Go-family of distribution proposed by Alizadeh *et al.* [2]. The GoLom distribution has been proposed by Oguntunde *et al.* [3] and statistical properties are carefully monitored.

Some of the modified and expanded forms of the GoLom have been studied such as; Gompertz Fréchet distribution [4], Power Gompertz distribution [5], Lomax Gompertz distribution [6], Gompertz-Alpha Power Inverted Exponential distribution [7], Gompertz Flexible Weibull distribution [8] Lomax Gompertz Makeham distribution [9], Gompertz Inverse Exponential distribution [10], Gompertz Burr Type XII distribution [11], Odd Lindley Gompertz distribution [12], Gompertz Length Biased Exponential distribution [13], and Gompertz Exponential distribution [14].

As a follow-up to the above fact, [15] suggested and investigated a newly compound distribution named "Gompertz Lomax distribution (GoLomD)" with four parameters. Based on various applications of the models to twelve real-life datasets [15], this distribution was shown to be skewed and flexible with varied forms, and performed better than the Weibull Lomax distribution [16], Beta Lomax distribution [17], and Kumaraswamy Lomax distribution [18].

From [15], the probability density function (PDF), the cumulative distribution (CDF), and the quantile function (QF) of the Gompertz Lomax (GoLom) distribution are respectively defined as

$$f(x) = \varphi\alpha\beta(1+\beta x)^{\alpha\gamma-1}e^{\left(\frac{\varphi}{\gamma}\right)[1-(1+\beta x)^{\gamma\alpha}]}, \gamma > 0, \alpha > 0, \beta > 0, \varphi > 0 \tag{1}$$

$$F(x) = 1 - e^{\left(\frac{\varphi}{\gamma}\right)[1-(1+\beta x)^{\gamma\alpha}]}, \gamma > 0, \alpha > 0, \beta > 0, \varphi > 0 \tag{2}$$

and

$$Q(u) = F^{-1}(u) = \beta^{-1}\left[\left\{1 - \frac{\gamma}{\varphi}log(1-u)\right\}^{1/\gamma\alpha} - 1\right], \tag{3}$$

where α, γ, and φ are the shape parameters and β is the scale parameter.

Figure 1 displays a graphical depiction of the Gompertz Lomax distribution's Pdf and Cdf for some given parameter values.

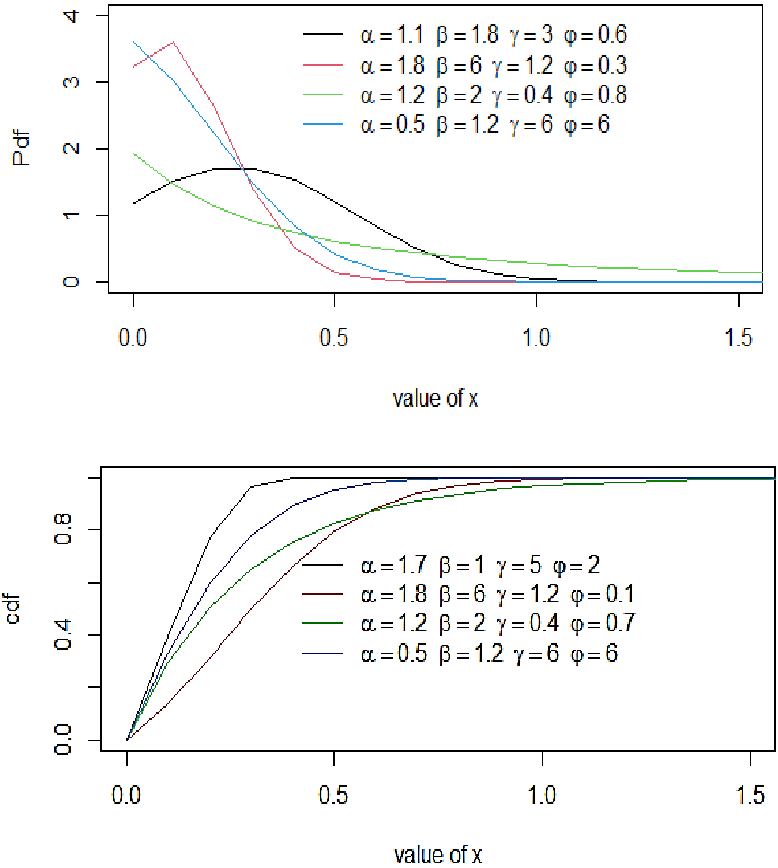

Figure 1: The plot of the PDF and CDF of the GoLom Distribution for Given Parameter Values

The objective of this paper is to determine a shape parameter of the GoLomD using only a Bayesian method with gamma, exponential, and Jeffreys's priors under seven loss functions. Following this opening part, the remainder of this paper's contents are given as follows: The MLE for the Shape parameter is produced in section 2. Bayes estimators depending on various loss functions under the assumption of gamma, exponential, and Jeffreys's priors are obtained in section 3. In section 4, The Bayesian estimations are compared by using the mean square error (MSE) of different sample sizes. Finally, in section 5 comments and conclusion are given.

2. MAXIMUM LIKELIHOOD ESTIMATION

Let $x_1, x_2, \ldots, x_n$ be a random sample of size n randomly distributed random variables with probability density functions that are independently and identically distributed. The likelihood is the data's joint probability function, but it is seen as a function of the parameters, with the observed data treated as a fixed variable quantity. Given that the values of x are acquired independently from a GoLom with unknown parameters, the likelihood function of GoLom is defined as follows:

$$L(x,\alpha,\beta,\gamma,\varphi) = \prod_{i=1}^{n} f(x,\alpha,\beta,\gamma,\varphi) \tag{4}$$

The likelihood function based on the GoLom Distribution is

$$L(x,\alpha,\beta,\gamma,\varphi) = (\varphi\alpha\beta)^n \sum_{i=1}^{n} (1+\beta x)^{\gamma\alpha-1}\, e^{\left(\frac{\varphi}{\gamma}\right)\sum_{i=1}^{n}[1-(1+\beta x)^{\gamma\alpha}]} \tag{5}$$

where l represents the log-likelihood function; hence

$$\begin{aligned} l = {} & n\log(\varphi) + n\log(\alpha) + n\log(\beta) \\ & + (\alpha\gamma - 1)\sum_{i=1}^{n}(1+\beta x_i) + \left(\frac{\varphi}{\gamma}\right)\sum_{i=1}^{n}[1-(1+\beta x_i)^{\gamma\alpha}] \end{aligned} \tag{6}$$

Differentiating w.r.t "$\boldsymbol{\varphi}$"

$$\frac{\partial \boldsymbol{l}}{\partial \boldsymbol{\varphi}} = \frac{\boldsymbol{n}}{\boldsymbol{\varphi}} + \left(\frac{\mathbf{1}}{\boldsymbol{\gamma}}\right)\sum_{\boldsymbol{i}=\mathbf{1}}^{\boldsymbol{n}}[\mathbf{1} - (\mathbf{1} + \boldsymbol{\beta x_i})^{\boldsymbol{\gamma\alpha}}].$$

Equating to zero

$$\frac{\partial l}{\partial \varphi} = 0$$

$$\frac{n}{\varphi} + \left(\frac{1}{\gamma}\right)\sum_{i=1}^{n}[1-(1+\beta x_i)^{\gamma\alpha}] = 0$$

$$\varphi^* = -\left[\frac{(n)(\gamma)}{\sum_{i=1}^{n}\{1-(1+\beta x_i)^{\gamma\alpha}\}}\right] \tag{7}$$

where $\boldsymbol{\varphi}^*$ is the MLE estimator of $\boldsymbol{\varphi}$. The MLEs of the other three parameters $\boldsymbol{\alpha}, \boldsymbol{\beta}$, and $\boldsymbol{\gamma}$ of GoLom distribution can be estimated simultaneously using $\frac{\partial \boldsymbol{l}}{\partial \boldsymbol{\alpha}} = \mathbf{0}$, $\frac{\partial \boldsymbol{l}}{\partial \boldsymbol{\beta}} = \mathbf{0}$, and $\frac{\partial \boldsymbol{l}}{\partial \boldsymbol{\gamma}} = \mathbf{0}$ respectively in [15].

3. BAYESIAN ESTIMATION

a) The gamma prior is defined as:

The Informative Prior (IP) of φ is gamma distribution with hyper-parameters a and b is given

$$P(\varphi) = \frac{b^a}{\Gamma(a)}\varphi^{a-1}e^{-b\varphi}\,;\ a,b > 0, \varphi > 0 \tag{8}$$

b) The exponential is defined as:

The prior distribution of φ is an exponential distribution with hyperparameters b is given

$$P(\varphi) = be^{-b\varphi}\,;\ b > 0, \varphi > 0 \tag{9}$$

c) Similarly, Jeffreys's prior is defined as:
The non-informative prior of φ is a uniform distribution

$$P(\varphi) = \propto \frac{1}{\varphi}; 0 < \varphi < \infty \tag{10}$$

1) Square Error Loss Function (SELF)
The square error loss function is defined from [44] as

$$L(\varphi, \phi^*) = (\varphi - \phi^*)^2 \tag{11}$$

where ϕ^* is an estimated value of φ.

2) Quadratic Loss Function (QLF)
The QLF is defined from [45] as

$$L(\varphi, \phi^*) = \left[1 - \frac{\phi^*}{\varphi}\right]^2 \tag{12}$$

where ϕ^* is an estimated value of φ.

3) Weighted Square Error Loss Function (WSELF)
The weighted SELF is defined from [46] as

$$L(\varphi, \phi^*) = \frac{(\varphi - \phi^*)^2}{\varphi} \tag{13}$$

where ϕ^* is an estimated value of φ.

4) Precautionary Loss Function (PLF)
The PLF is defined from [47] as

$$L(\varphi, \phi^*) = \frac{(\varphi - \phi^*)^2}{\phi^*} \tag{14}$$

where ϕ^* is an estimated value of φ.

5) K Loss Function (KLF)
The KLF is given from [48] as

$$L(\varphi, \phi^*) = \left[\sqrt{\frac{\phi^*}{\varphi}} - \sqrt{\frac{\varphi}{\phi^*}}\right]^2 \tag{15}$$

where ϕ^* is an estimated value of φ.

6) Degroot Loss Function (DLF)
The DeGroot LF is given from [49] as

$$L(\varphi, \phi^*) = \left[\frac{(\varphi - \phi^*)}{\phi^*}\right]^2 \tag{16}$$

where ϕ^* is an estimated value of φ.

7) Al-Bayyati's Loss Function (ABLF)

The ABLF is defined from [50] as

$$L(\varphi, \phi^*) = \varphi^c (\phi^* - \varphi)^2 \,, c \in \mathbb{R}^+ \tag{17}$$

where ϕ^* is an estimated value of φ.

d) Posterior Distribution

Let X be a random sample with n observations then the posterior distribution for unknown parameter φ is

$$P\left({}^{\varphi}/_{x}\right) \propto P(\varphi) \times L(x) \tag{18}$$

where $P(\varphi)$ and $L(x)$ are the prior distribution and the likelihood function, respectively.

3.1 Bayesian Analysis under Gamma Prior with Seven Loss Functions

The posterior distribution of φ, assuming a gamma prior distribution, is computed as follows from equation (18) using the integration through substitution technique:

$$P\left({}^{\varphi}/_{x}\right) \propto \frac{b^a}{\Gamma(a)} \varphi^{a-1} e^{-b\varphi} (\varphi\alpha\beta)^n \sum_{i=1}^{n} (1+\beta x_i)^{\gamma\alpha-1} \, e^{\left(\frac{\varphi}{\gamma}\right)\sum_{i=1}^{n}[1-(1+\beta x_i)^{\gamma\alpha}]}$$

$$P\left({}^{\varphi}/_{x}\right) \propto \varphi^{a+n-1} e^{-b\varphi} e^{\left(\frac{\varphi}{\gamma}\right)\sum_{i=1}^{n}[1-(1+\beta x_i)^{\gamma\alpha}]}$$

$$P\left({}^{\varphi}/_{x}\right) \propto K \varphi^{(a+n)-1} e^{-\varphi\left[b-\frac{1}{\gamma}\sum_{i=1}^{n}\{1-(1+\beta x_i)^{\gamma\alpha}\}\right]}$$

$$P\left({}^{\varphi}/_{x}\right) = \frac{\left(b-\frac{1}{\gamma}\sum_{i=1}^{n}\{1-(1+\beta x_i)^{\gamma\alpha}\}\right)^{a+n}}{\Gamma(a+n)} \varphi^{a+n-1} e^{-\varphi\left(b-\frac{1}{\gamma}\sum_{i=1}^{n}\{1-(1+\beta x_i)^{\gamma\alpha}\}\right)} \tag{19}$$

where $K = \frac{\left(b-\frac{1}{\gamma}\sum_{i=1}^{n}\{1-(1+\beta x_i)^{\gamma\alpha}\}\right)^{a+n}}{\Gamma(a+n)}$

$$\therefore P\left({}^{\varphi}/_{x}\right) \sim Gamma\left(a+n, b - \frac{1}{\gamma}\sum_{i=1}^{n}\{1-(1+\beta x_i)^{\gamma\alpha}\}\right)$$

As a result, the Bayes estimators under gamma prior, to utilizing SELF, QLF, WSELF, PLF, KLF, DLF, and ABLF are as follows:

$$\phi^*_{SELF} = E_{\varphi/x}(\varphi) = \int_0^\infty \varphi P\left({}^{\varphi}/_{x}\right) d\varphi = \frac{(a+n)}{b-\frac{1}{\gamma}\sum_{i=1}^{n}\{1-(1+\beta x_i)^{\gamma\alpha}\}} \tag{20}$$

$$\phi^*_{QLF} = \frac{E_{\varphi/x}(\varphi^{-1})}{E_{\varphi/x}(\varphi^{-2})} = \frac{\int_0^\infty \varphi^{-1} P\left({}^{\varphi}/_{x}\right) d\varphi}{\int_0^\infty \varphi^{-2} P\left({}^{\varphi}/_{x}\right) d\varphi} = \frac{(a+n-2)}{b-\frac{1}{\gamma}\sum_{i=1}^{n}\{1-(1+\beta x_i)^{\gamma\alpha}\}} \tag{21}$$

$$\phi^*_{WSELF} = \left(E_{\varphi/x}(\varphi^{-1})\right)^{-1} = \left(\int_0^\infty \varphi^{-1} P\left({}^{\varphi}/_{x}\right) d\varphi\right)^{-1}$$
$$= \frac{(a+n-1)}{b - \frac{1}{\gamma}\sum_{i=1}^{n}\{1-(1+\beta x_i)^{\gamma\alpha}\}} \tag{22}$$

$$\phi^*_{PLF} = \sqrt{E_{\varphi/x}(\varphi^2)} = \sqrt{\int_0^\infty \varphi^2 P\left({}^{\varphi}/_{x}\right) d\varphi}$$
$$= \sqrt{\frac{(a+n)(a+n+1)}{\left(b - \frac{1}{\gamma}\sum_{i=1}^{n}\{1-(1+\beta x_i)^{\gamma\alpha}\}\right)^2}} \tag{23}$$

$$\phi^*_{KLF} = \sqrt{\frac{E_{\varphi/x}(\varphi)}{E_{\varphi/x}(\varphi^{-1})}} = \sqrt{\frac{\int_0^\infty \varphi P\left({}^{\varphi}/_{x}\right) d\varphi}{\int_0^\infty \varphi^{-1} P\left({}^{\varphi}/_{x}\right) d\varphi}}$$
$$= \sqrt{\frac{(a+n)(a+n-1)}{\left(b - \frac{1}{\gamma}\sum_{i=1}^{n}\{1-(1+\beta x_i)^{\gamma\alpha}\}\right)^2}} \tag{24}$$

$$\phi^*_{DLF} = \frac{E_{\varphi/x}(\varphi^2)}{E_{\varphi/x}(\varphi)} = \frac{\int_0^\infty \varphi^2 P\left({}^{\varphi}/_{x}\right) d\varphi}{\int_0^\infty \varphi P\left({}^{\varphi}/_{x}\right) d\varphi} = \frac{(a+n+1)}{\left(b - \frac{1}{\gamma}\sum_{i=1}^{n}\{1-(1+\beta x_i)^{\gamma\alpha}\}\right)} \tag{25}$$

$$\phi^*_{ABLF} = \frac{E_{\varphi/x}(\varphi^{c+1})}{E_{\varphi/x}(\varphi^{c})} = \frac{\int_0^\infty \varphi^{c+1} P\left({}^{\varphi}/_{x}\right) d\varphi}{\int_0^\infty \varphi^{c} P\left({}^{\varphi}/_{x}\right) d\varphi}$$
$$= \frac{\Gamma(a+n+c+1)}{\left(b - \frac{1}{\gamma}\sum_{i=1}^{n}\{1-(1+\beta x_i)^{\gamma\alpha}\}\right)\Gamma(a+n+c)} \tag{26}$$

3.2 Bayesian analysis under exponential prior with seven loss functions

The posterior distribution of φ, assuming an exponential prior distribution, is computed as follows from equation (18) using the integration through replacement technique:

$$P\left({}^{\varphi}/_{x}\right) \propto be^{-b\varphi}(\varphi\alpha\beta)^n \sum_{i=1}^{n}(1+\beta x_i)^{\gamma\alpha-1}\, e^{\left(\frac{\varphi}{\gamma}\right)\sum_{i=1}^{n}[1-(1+\beta x_i)^{\gamma\alpha}]}$$

$$P\left({}^{\varphi}/_{x}\right) \propto \varphi^n e^{-b\varphi} e^{\left(\frac{\varphi}{\gamma}\right)\sum_{i=1}^{n}[1-(1+\beta x_i)^{\gamma\alpha}]}$$

$$P\left({}^{\varphi}/_{x}\right) \propto K\varphi^{(n+1)-1} e^{-\varphi\left[b-\frac{1}{\gamma}\sum_{i=1}^{n}\{1-(1+\beta x_i)^{\gamma\alpha}\}\right]}$$

$$P\left(\varphi/_x\right) = \frac{\left(b - \frac{1}{\gamma}\sum_{i=1}^{n}\{1-(1+\beta x_i)^{\gamma\alpha}\}\right)^{a+1}}{\Gamma(a+1)} \varphi^{(a+1)-1} e^{-\varphi\left(b-\frac{1}{\gamma}\sum_{i=1}^{n}\{1-(1+\beta x_i)^{\gamma\alpha}\}\right)} \mathrm{m} \tag{27}$$

where $K = \frac{\left(b-\frac{1}{\gamma}\sum_{i=1}^{n}\{1-(1+\beta x_i)^{\gamma\alpha}\}\right)^{a+1}}{\Gamma(a+1)}$

$$\therefore P\left(\varphi/_x\right) \sim Gamma\left(a+1, b - \frac{1}{\gamma}\sum_{i=1}^{n}\{1-(1+\beta x_i)^{\gamma\alpha}\}\right)$$

As a result, the Bayes estimators under exponential prior utilizing SELF, QLF, WSELF, PLF, KLF, DLF, and ABLF are as follows:

$$\phi^*{}_{SELF} = E_{\varphi/_x}(\varphi) = \int_0^\infty \varphi P\left(\varphi/_x\right) d\varphi = \frac{(a+1)}{b - \frac{1}{\gamma}\sum_{i=1}^{n}\{1-(1+\beta x_i)^{\gamma\alpha}\}} \tag{28}$$

$$\phi^*{}_{QLF} = \frac{E_{\varphi/_x}(\varphi^{-1})}{E_{\varphi/_x}(\varphi^{-2})} = \frac{\int_0^\infty \varphi^{-1} P\left(\varphi/_x\right) d\varphi}{\int_0^\infty \varphi^{-2} P\left(\varphi/_x\right) d\varphi} = \frac{(a+1-2)}{b - \frac{1}{\gamma}\sum_{i=1}^{n}\{1-(1+\beta x_i)^{\gamma\alpha}\}} \tag{29}$$

$$\begin{aligned}\phi^*{}_{WSELF} &= \left(E_{\varphi/_x}(\varphi^{-1})\right)^{-1} = \left(\int_0^\infty \varphi^{-1} P\left(\varphi/_x\right) d\varphi\right)^{-1} \\ &= \frac{(a+1-1)}{b - \frac{1}{\gamma}\sum_{i=1}^{n}\{1-(1+\beta x_i)^{\gamma\alpha}\}}\end{aligned} \tag{30}$$

$$\begin{aligned}\phi^*{}_{PLF} &= \sqrt{E_{\varphi/_x}(\varphi^2)} = \sqrt{\int_0^\infty \varphi^2 P\left(\varphi/_x\right) d\varphi} \\ &= \sqrt{\frac{(a+1)(a+1+1)}{\left(b - \frac{1}{\gamma}\sum_{i=1}^{n}\{1-(1+\beta x_i)^{\gamma\alpha}\}\right)^2}}\end{aligned} \tag{31}$$

$$\begin{aligned}\phi^*{}_{KLF} &= \sqrt{\frac{E_{\varphi/_x}(\varphi)}{E_{\varphi/_x}(\varphi^{-1})}} = \sqrt{\frac{\int_0^\infty \varphi P\left(\varphi/_x\right) d\varphi}{\int_0^\infty \varphi^{-1} P\left(\varphi/_x\right) d\varphi}} \\ &= \sqrt{\frac{(a+1)(a+1-1)}{\left(b - \frac{1}{\gamma}\sum_{i=1}^{n}\{1-(1+\beta x_i)^{\gamma\alpha}\}\right)^2}}\end{aligned} \tag{32}$$

$$\phi^*_{DLF} = \frac{E_{\varphi/x}(\varphi^2)}{E_{\varphi/x}(\varphi)} = \frac{\int_0^\infty \varphi^2 P(\varphi/x) d\varphi}{\int_0^\infty \varphi P(\varphi/x) d\varphi} = \frac{(a+1+1)}{\left(b - \frac{1}{\gamma}\sum_{i=1}^n \{1-(1+\beta x_i)^{\gamma\alpha}\}\right)} \quad (33)$$

$$\phi^*_{ABLF} = \frac{E_{\varphi/x}(\varphi^{c+1})}{E_{\varphi/x}(\varphi^c)} = \frac{\int_0^\infty \varphi^{c+1} P(\varphi/x) d\varphi}{\int_0^\infty \varphi^c P(\varphi/x) d\varphi} = \frac{\Gamma(a+1+c+1)}{\left(b - \frac{1}{\gamma}\sum_{i=1}^n \{1-(1+\beta x_i)^{\gamma\alpha}\}\right)\Gamma(a+1+c)} \quad (34)$$

3.3 Bayesian analysis under Jeffreys's prior with seven loss functions

The posterior distribution, assuming Jeffreys's prior distribution, is computed as follows from equation (18) using the integration through replacement technique:

$$P(\varphi/x) \propto \frac{1}{\varphi}(\varphi\alpha\beta)^n \sum_{i=1}^n (1+\beta x_i)^{\gamma\alpha-1} e^{\left(\frac{\varphi}{\gamma}\right)\sum_{i=1}^n [1-(1+\beta x_i)^{\gamma\alpha}]}$$

$$P(\varphi/x) \propto \varphi^{n-1} e^{\left(\frac{\varphi}{\gamma}\right)\sum_{i=1}^n [1-(1+\beta x_i)^{\gamma\alpha}]}$$

$$P(\varphi/x) \propto K\varphi^{(n)-1} e^{-\varphi\left[-\frac{1}{\gamma}\sum_{i=1}^n \{1-(1+\beta x_i)^{\gamma\alpha}\}\right]}$$

$$P(\varphi/x) = \frac{\left(-\frac{1}{\gamma}\sum_{i=1}^n \{1-(1+\beta x_i)^{\gamma\alpha}\}\right)^n}{\Gamma(n)} \varphi^{n-1} e^{-\varphi\left(-\frac{1}{\gamma}\sum_{i=1}^n \{1-(1+\beta x_i)^{\gamma\alpha}\}\right)} \quad (35)$$

where $K = \frac{\left(-\frac{1}{\gamma}\sum_{i=1}^n \{1-(1+\beta x_i)^{\gamma\alpha}\}\right)^n}{\Gamma(n)}$

$$\therefore P(\varphi/x) \sim Gamma\left(n, -\frac{1}{\gamma}\sum_{i=1}^n \{1-(1+\beta x_i)^{\gamma\alpha}\}\right)$$

As a result, the Bayes estimators under Jeffreys' prior utilizing SELF, QLF, WSELF, PLF, KLF, DLF, and ABLF are as follows:

$$\phi^*_{SELF} = E_{\varphi/x}(\varphi) = \int_0^\infty \varphi P(\varphi/x) d\varphi = \frac{(n)}{-\frac{1}{\gamma}\sum_{i=1}^n \{1-(1+\beta x_i)^{\gamma\alpha}\}} \quad (36)$$

$$\phi^*_{QLF} = \frac{E_{\varphi/x}(\varphi^{-1})}{E_{\varphi/x}(\varphi^{-2})} = \frac{\int_0^\infty \varphi^{-1} P(\varphi/x) d\varphi}{\int_0^\infty \varphi^{-2} P(\varphi/x) d\varphi} = \frac{(n-2)}{-\frac{1}{\gamma}\sum_{i=1}^n \{1-(1+\beta x_i)^{\gamma\alpha}\}} \quad (37)$$

$$\phi^*_{WSELF} = \left(E_{\varphi/x}(\varphi^{-1})\right)^{-1} = \left(\int_0^\infty \varphi^{-1} P(\varphi/x) d\varphi\right)^{-1} = \frac{(n-1)}{-\frac{1}{\gamma}\sum_{i=1}^n \{1-(1+\beta x_i)^{\gamma\alpha}\}} \quad (38)$$

$$\phi^*{}_{PLF} = \sqrt{E\varphi_{/x}(\varphi^2)} = \sqrt{\int_0^\infty \varphi^2 P\left({}^{\varphi}/_{x}\right) d\varphi}$$
$$= \sqrt{\frac{(n)(n+1)}{\left(-\frac{1}{\gamma}\sum_{i=1}^{n}\{1-(1+\beta x_i)^{\gamma\alpha}\}\right)^2}} \tag{39}$$

$$\phi^*{}_{KLF} = \sqrt{\frac{E\varphi_{/x}(\varphi)}{E\varphi_{/x}(\varphi^{-1})}} = \sqrt{\frac{\int_0^\infty \varphi P\left({}^{\varphi}/_{x}\right) d\varphi}{\int_0^\infty \varphi^{-1} P\left({}^{\varphi}/_{x}\right) d\varphi}}$$
$$= \sqrt{\frac{(n)(n-1)}{\left(-\frac{1}{\gamma}\sum_{i=1}^{n}\{1-(1+\beta x_i)^{\gamma\alpha}\}\right)^2}} \tag{40}$$

$$\phi^*{}_{DLF} = \frac{E\varphi_{/x}(\varphi^2)}{E\varphi_{/x}(\varphi)} = \frac{\int_0^\infty \varphi^2 P\left({}^{\varphi}/_{x}\right) d\varphi}{\int_0^\infty \varphi P\left({}^{\varphi}/_{x}\right) d\varphi} = \frac{(n+1)}{\left(-\frac{1}{\gamma}\sum_{i=1}^{n}\{1-(1+\beta x_i)^{\gamma\alpha}\}\right)} \tag{41}$$

$$\phi^*{}_{ABLF} = \frac{E\varphi_{/x}(\varphi^{c+1})}{E\varphi_{/x}(\varphi^{c})} = \frac{\int_0^\infty \varphi^{c+1} P\left({}^{\varphi}/_{x}\right) d\varphi}{\int_0^\infty \varphi^{c} P\left({}^{\varphi}/_{x}\right) d\varphi}$$
$$= \frac{\Gamma(n+c+1)}{\left(-\frac{1}{\gamma}\sum_{i=1}^{n}\{1-(1+\beta x_i)^{\gamma\alpha}\}\right)\Gamma(n+c)}. \tag{42}$$

4. SIMULATION

To compare the estimators $\phi^*{}_{SELF}$, $\phi^*{}_{QLF}$, $\phi^*{}_{WSELF}$, $\phi^*{}_{PLF}$, $\phi^*{}_{KLF}$, $\phi^*{}_{DLF}$ and $\phi^*{}_{ABLF}$, we have evaluated the estimators' mean square errors (MSE). The estimator's MSE ϕ^* is defined as

$$MSE = E(\varphi - \phi^*)^2 = Var(\phi^*) + [Bias(\phi^*)]^2$$

The sample sizes have been chosen in this simulation research $n = 10, 20, 30, 40$ and 50 for several values of $a = 1$; $b = 2$; $\alpha = 1$; $\beta = 1$; $\gamma = 1$;$\varphi = 3, 2.5, 1, 0.5, 0.1$; $c = 0.1$,$b = 1$; $\alpha = 1$; $\beta = 1$; $\gamma = 1$; $\varphi = 1.7, 1.3, 0.9, 0.5, 0.1$; $c = 1$, and $\alpha = 1$; $\beta = 1$; $\gamma = 1$; $\varphi = 2, 1.6, 0.8, 0.4, 0.1$; $c = 0.1$ based on simulation ($S = 10{,}000$). Again, using the R-Codes to determine the estimates, MSE, as well as Bayes estimators of the GoLom Distribution. The results, as well as their diagrams (created in Microsoft Excel), are given under different priors.

Table 1
Estimations and MSE (within Parenthesis) for ϕ^* under Gamma Prior at $S = 10{,}000$, $a = 1$, $b = 2$, $\alpha = 1$, $\beta = 1$, $\gamma = 1$, $c = 0.1$

n	Parameters	Methods of Estimation							
	φ	φ^*_{MLE}	ϕ^*_{SELF}	ϕ^*_{QLF}	ϕ^*_{WSELF}	ϕ^*_{PLF}	ϕ^*_{KLF}	ϕ^*_{DLF}	ϕ^*_{ABLF}
10	3	2.4679 (1.4457)	1.6250 (0.8969)	1.6954 (1.6627)	2.5584 (1.2447)	3.1731 (0.7600)	2.0597 (1.0677)	2.0775 (0.6429)	2.8950 (2.2008)
	2.5	2.4106 (1.0638)	1.2596 (0.4963)	1.4679 (0.9801)	1.6878 (0.7056)	1.7817 (0.4246)	1.8300 (0.5940)	2.1247 (0.3597)	1.6191 (1.5109)
	1	0.8731 (0.1618)	0.8415 (0.0717)	0.9357 (0.0868)	0.6839 (0.0715)	1.1679 (0.0791)	0.7610 (0.0695)	1.3399 (0.0926)	2.2090 (0.2461)
	0.5	0.4266 (0.0399)	0.5808 (0.0283)	0.3082 (0.0216)	0.3384 (0.0227)	0.6875 (0.0340)	0.5683 (0.0256)	0.4712 (0.0430)	0.7988 (0.0618)
	0.1	0.0999 (0.0016)	0.1332 (0.0020)	0.1257 (0.0011)	0.1053 (0.0014)	0.0744 (0.0024)	0.1045 (0.0017)	0.1193 (0.0027)	0.0824 (0.0023)
20	3	2.1914 (0.5857)	2.2023 (0.4510)	2.6035 (0.7033)	2.1461 (0.5631)	2.5120 (0.3967)	3.1087 (0.4980)	3.1567 (0.3533)	2.6462 (0.7409)
	2.5	3.3208 (0.3984)	1.7905 (0.2681)	1.4460 (0.4111)	2.3211 (0.3322)	2.6622 (0.2400)	1.9026 (0.2948)	2.7821 (0.2204)	1.5310 (0.5102)
	1	1.1469 (0.0663)	1.0217 (0.0443)	0.8740 (0.0454)	0.9950 (0.0427)	1.0692 (0.0457)	0.8096 (0.0424)	1.1078 (0.0499)	1.2095 (0.0790)
	0.5	0.3175 (0.0160)	0.5504 (0.0141)	0.5116 (0.0114)	0.4857 (0.0117)	0.3345 (0.0157)	0.4709 (0.0127)	0.5888 (0.0166)	0.5929 (0.0196)
	0.1	0.0936 (0.0006)	0.0777 (0.0007)	0.0839 (0.0005)	0.1219 (0.0006)	0.1525 (0.0008)	0.1564 (0.0006)	0.1046 (0.0009)	0.0826 (0.0008)
30	3	3.4463 (0.3523)	2.9684 (0.2905)	1.9807 (0.4226)	2.2060 (0.3467)	2.8746 (0.2668)	2.5931 (0.3182)	3.3405 (0.2494)	3.1007 (0.4091
	2.5	3.1563 (0.2424)	3.4236 (0.1824)	1.9298 (0.2554)	1.6464 (0.2147)	2.4794 (0.1705)	2.0582 (0.2007)	3.0582 (0.1640)	2.2578 (0.2950)
	1	1.0055 (0.0405)	1.1147 (0.0308)	0.8391 (0.0311)	1.0785 (0.0303)	1.0663 (0.0312)	0.7066 (0.0298)	1.3070 (0.0340)	1.0837 (0.0475)
	0.5	0.8069 (0.0100)	0.4919 (0.0090)	0.4581 (0.0079)	0.4989 (0.0080)	0.5046 (0.0098)	0.5973 (0.0086)	0.5162 (0.0106)	0.4414 (0.0115)
	0.1	0.1147 (0.0003)	0.0852 (0.0004)	0.0777 (0.0003)	0.0973 (0.0003)	0.0896 (0.0004)	0.0960 (0.0004)	0.0957 (0.0099)	0.1358 (0.0004)
40	3	2.5567 (0.2580)	3.2711 (0.2212)	2.3940 (0.2924)	2.7179 (0.2618)	2.6118 (0.2110)	2.5084 (0.2339)	2.6065 (0.1969)	2.7814 (0.2821)
	2.5	2.4279 (0.1740)	1.9707 (0.1370)	2.0075 (0.1815)	2.4561 (0.1610)	1.7168 (0.1358)	2.0057 (0.1487)	2.2788 (0.1317)	2.2924 (0.1974)
	1	0.9765 (0.0277)	0.8666 (0.0235)	0.9670 (0.0236)	0.8598 (0.0236)	0.7754 (0.0243)	0.9328 (0.0229)	1.1559 (0.0261)	0.7044 (0.0322)
	0.5	0.5616 (0.0069)	0.5243 (0.0065)	0.4542 (0.0061)	0.4366 (0.0063)	0.4567 (0.0071)	0.4149 (0.0065)	0.5430 (0.0077)	0.3964 (0.0080)
	0.1	0.1014 (0.0002)	0.1071 (0.0003)	0.1172 (0.0002)	0.0867 (0.0002)	0.1139 (0.0003)	0.1289 (0.0002)	0.0922 (0.0003)	0.1068 (0.0003)
50	3	2.3888 (0.1936)	2.5261 (0.1712)	2.5156 (0.2280)	2.2892 (0.1962)	2.5012 (0.1686)	2.5237 (0.1841)	3.2166 (0.1621)	3.7567 (0.2229)
	2.5	2.4225 (0.1396)	2.3412 (0.1143)	1.8484 (0.1442)	1.9924 (0.1264)	2.3379 (0.1096)	2.1630 (0.1181)	2.4228 (0.1053)	2.5288 (0.1485)
	1	0.8667 (0.0217)	1.0100 (0.0192)	1.3319 (0.0188)	1.0820 (0.0190)	0.9683 (0.0192)	0.9736 (0.0191)	0.8826 (0.0201)	0.7506 (0.0246)
	0.5	0.5276 (0.0055)	0.6443 (0.0052)	0.4904 (0.0049)	0.3705 (0.0049)	0.5751 (0.0054)	0.4485 (0.0051)	0.5122 (0.0056)	0.4903 (0.0061)
	0.1	0.1021 (0.0002)	0.0974 (0.0002)	0.0844 (0.0002)	0.0982 (0.0002)	0.1100 (0.0002)	0.1081 (0.0002)	0.1130 (0.0002)	0.0952 (0.0002)

From Table 1, it has been noticed that the MSE of ABLF is quite high for such a tiny sample size but with increasing sample size, they drop drastically and become more comparable to other estimators. Among Bayes estimator, DLF gives the lower MSE value, when the sample size is less but they're almost equal when compared to a large sample.

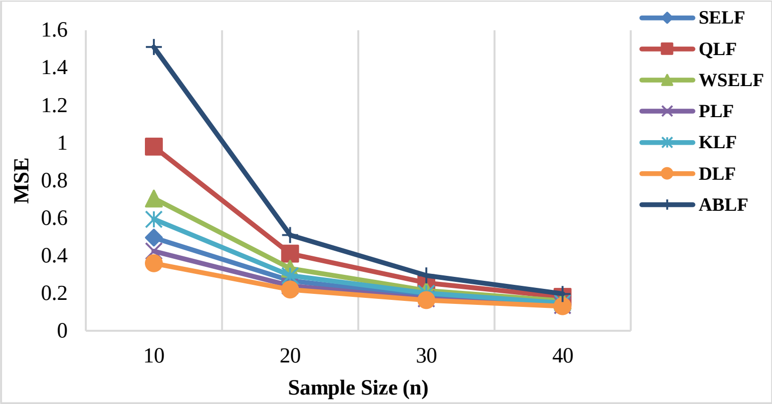

Figure 2: Graph of MSEs of Different Estimators of φ of Gompertz Lomax Distribution where $a = 1$, $b = 2$, $\alpha = 1$, $\beta = 1$, $\gamma = 1$, $\varphi = 3$ and $c = 0.1$

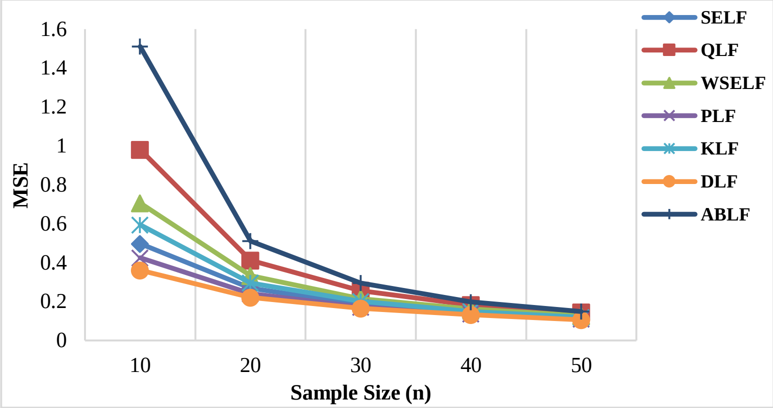

Figure 3: Graph of MSEs of Different Estimators of φ of Gompertz Lomax distribution where $a = 1$, $b = 2$, $\alpha = 1$, $\beta = 1$, $\gamma = 1$, $\varphi = 2.5$ and $c = 0.1$

Table 2
Estimations and MSE (within Parenthesis) for ϕ^* under Exponential Prior at $S = 10{,}000,\ b = 1,\ \alpha = 1,\ \beta = 1,\ \gamma = 1,\ c = 1$

n	Parameters	Methods of Estimation							
	φ	φ^*_{MLE}	ϕ^*_{SELF}	ϕ^*_{QLF}	ϕ^*_{WSELF}	ϕ^*_{PLF}	ϕ^*_{KLF}	ϕ^*_{DLF}	ϕ^*_{ABLF}
10	1.7	2.1342 (0.4931)	1.5199 (0.2389)	1.5994 (0.2411)	1.1573 (0.2155)	1.8906 (0.2685)	1.3924 (0.2168)	1.7066 (0.3180)	2.9579 (0.9591)
	1.3	0.7252 (0.2918)	1.9196 (0.1641)	0.9356 (0.1428)	1.0095 (0.1397)	2.6747 (0.2020)	1.2189 (0.1518)	2.0045 (0.2351)	1.2042 (0.5307)
	0.9	0.6101 (0.1327)	1.6220 (0.1047)	0.7827 (0.0728)	0.8115 (0.0779)	1.0486 (0.1184)	1.5795 (0.0880)	1.5300 (0.1382)	0.9701 (0.2752)
	0.5	0.3660 (0.0438)	0.7478 (0.0418)	0.3829 (0.0234)	0.3889 (0.0300)	0.6598 (0.0495)	0.6286 (0.0359)	0.4134 (0.0592)	0.5887 (0.0844)
	0.1	0.1317 (0.0016)	0.1658 (0.0021)	0.0479 (0.0011)	0.1519 (0.0015)	0.0779 (0.0025)	0.0966 (0.0018)	0.1038 (0.0030)	0.1265 (0.0033)
20	1.7	2.4617 (0.1859)	1.1854 (0.1319)	1.4527 (0.1318)	2.0865 (0.1267)	1.3083 (0.1415)	1.7510 (0.1319)	2.9038 (0.1663)	1.9634 (0.2924)
	1.3	1.3461 (0.1095)	0.9169 (0.0893)	1.6772 (0.0773)	1.2037 (0.0781)	0.8894 (0.0958)	1.2687 (0.0813)	1.2432 (0.1017)	1.2735 (0.1682)
	0.9	1.4952 (0.0501)	1.0785 (0.0489)	0.6683 (0.0383)	0.9388 (0.0399)	1.1206 (0.0519)	1.4115 (0.0436)	1.3928 (0.0573)	0.8074 (0.0812)
	0.5	0.7362 (0.0159)	0.4073 (0.0158)	0.4810 (0.0162)	0.5953 (0.0139)	0.4171 (0.0185)	0.4886 (0.0151)	0.4420 (0.0205)	0.6872 (0.02500
	0.1	0.0775 (0.0006)	0.0896 (0.0007)	0.0978 (0.0005)	0.0909 (0.0006)	0.0891 (0.0008)	0.1081 (0.0006)	0.1116 (0.0099)	0.1224 (0.0009)
30	1.7	1.7915 (0.1173)	1.6903 (0.0916)	1.4584 (0.0887)	1.6361 (0.0879)	2.4728 (0.0962)	1.8907 (0.0890)	1.8069 (0.1062)	2.1938 (0.1532)
	1.3	1.4479 (0.0663)	1.9404 (0.0584	0.8661 (0.0542)	1.5188 (0.0533)	1.2063 (0.0618)	1.5922 (0.0548)	1.2699 (0.0643)	1.4791 (0.0931)
	0.9	0.9108 (0.0308)	0.8696 (0.0308)	0.9854 (0.0255)	0.9637 (0.0267)	0.8964 (0.0320)	0.7329 (0.0287)	1.0517 (0.0347)	1.0329 (0.0442)
	0.5	0.5680 (0.0099)	0.5684 (0.0102)	0.6129 (0.0086)	0.6033 (0.0090)	0.4242 (0.0108)	0.4789 (0.0091)	0.4969 (0.0119)	0.5505 (0.01350
	0.1	0.0780 (0.0003)	0.0918 (0.0004)	0.0807 (0.0003)	0.0945 (0.0003)	0.0930 (0.0004)	0.1050 (0.0004)	0.1191 (0.0005)	0.1151 (0.0005)
40	1.7	1.6872 (0.0808)	1.7339 (0.0706)	1.6078 (0.0675)	1.3834) (0.0673)	1.9151 (0.0728)	1.7078 (0.0693)	1.5206 (0.0751)	1.5861 (0.1072)
	1.3	1.2633 (0.0484)	1.1771 (0.0426)	1.1894 (0.0407)	1.0644 (0.0402)	1.2780 (0.0467)	1.2819 (0.0417)	1.0782 (0.0475)	1.2655 (0.06330
	0.9	0.7395 (0.0228)	0.7016 (0.0227)	1.1189 (0.0196)	0.8428 (0.0198)	0.9089 (0.0232)	0.8280 (0.0211)	0.8918 (0.0247)	0.8635 (0.0303)
	0.5	0.5642 (0.0070)	0.6069 (0.0071)	0.5582 (0.0062)	0.4686 (0.0064)	0.4060 (0.0077)	0.5360 (0.0069)	0.5375 (0.0083)	0.5830 (0.0091)
	0.1	0.0942 (0.0002)	0.1095 (0.0003)	0.0880 (0.0002)	0.1006 (0.0002	0.1049 (0.0003)	0.0878 (0.0002)	0.0920 (0.0003)	0.0802 (0.0003)
50	1.7	1.7574 (0.0625)	1.5961 (0.0562)	1.3606 (0.0563)	1.2791 (0.05430	2.0006 (0.0587)	2.0890 (0.0554)	1.7828 (0.0610)	1.7887 (0.0803)
	1.3	1.4229 (0.0368)	1.4983 (0.0347)	1.1697 (0.0327)	1.2157 (0.0331)	1.3023 (0.0358)	1.1010 (0.0338)	1.3903 (0.0377)	1.5312 (0.0446)
	0.9	0.7247 (0.0182)	0.8109 (0.0179)	0.9348 (0.0162)	1.2084 (0.0163)	1.0677 (0.0177)	1.1840 (0.0171)	1.0511 (0.0184)	0.7117 (0.0219)
	0.5	0.5715 (0.0054)	0.4688 (0.0057)	0.4838 (0.0049)	0.5619 (0.0053)	0.4976 (0.0059)	0.5733 (0.0054)	0.4432 (0.0063)	0.4018 (0.0068)
	0.1	0.1113 (0.0002)	0.1006 (0.0002)	0.1055 (0.0002)	0.0885 (0.0002)	0.0911 (0.0002)	0.1236 (0.0002)	0.0982 (0.0002)	0.1008 (0.0002)

From Table 2, it has been noticed that the MSE of ABLF is quite high for such a tiny sample size but with increasing sample size, they drop drastically and become more comparable to other estimators. Among Bayes estimator, WSELF gives the lower MSE value, when the sample size is less but they're almost equal when compared to a large sample.

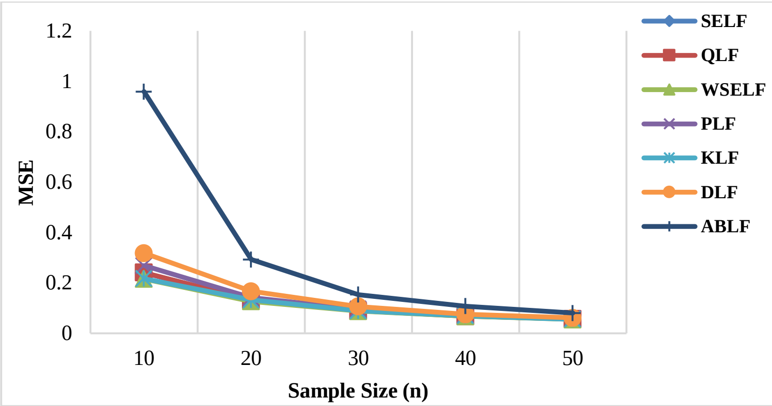

Figure 4: Graph of MSEs of Different Estimators of φ of Gompertz Lomax Distribution where $b = 1, \alpha = 1, \beta = 1, \gamma = 1, \varphi = 1.7$ and $c = 1$

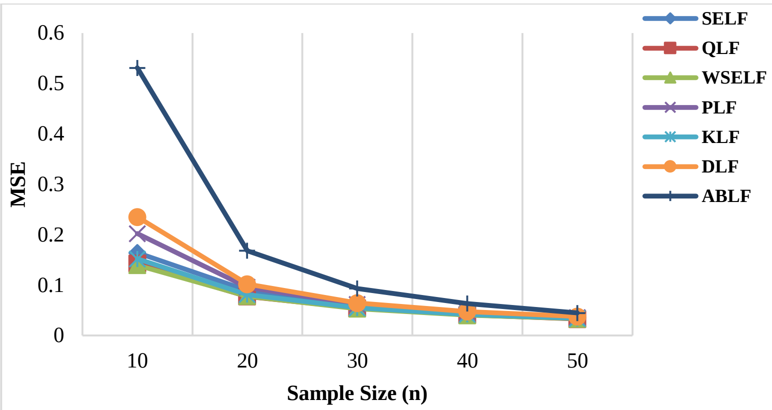

Figure 5: Graph of MSEs of Different Estimators of φ of Gompertz Lomax Distribution where $b = 1, \alpha = 1, \beta = 1, \gamma = 1, \varphi = 1.3$ and $c = 1$

Table 3
Estimations and MSE (within Parenthesis) for ϕ^* under Jeffreys's Prior at $S = 10{,}000$, $\alpha = 1$, $\beta = 1$, $\gamma = 1$, $c = 0.1$

n	Parameters	Methods of Estimation							
	φ	φ^*_{MLE}	ϕ^*_{SELF}	ϕ^*_{QLF}	ϕ^*_{WSELF}	ϕ^*_{PLF}	ϕ^*_{KLF}	ϕ^*_{DLF}	ϕ^*_{ABLF}
10	2.0	1.6099 (0.6760)	2.1774 (0.6770)	2.0282 (0.4250)	2.1431 (0.5074)	2.9012 (0.8073)	2.8717 (0.5964)	3.6726 (0.9431)	2.5479 (0.6672)
	1.6	1.3108 (0.4462)	1.0984 (0.4184)	0.8369 (0.2790)	1.3148 (0.3145)	2.1758 (0.4941)	1.2653 (0.3538)	1.1415 (0.6106)	1.9759 (0.4520)
	0.8	0.9918 (0.1092)	0.8528 (0.1099)	0.7121 (0.0693)	1.3408 (0.0812)	0.5692 (0.1230)	0.5782 (0.0866)	1.0554 (0.1569)	0.7750 (0.1100)
	0.4	0.3602 (0.0257)	0.6190 (0.0271)	0.2931 (0.0177)	0.3741 (0.0206)	0.7216 (0.0300)	0.3017 (0.0233)	0.4780 (0.0371)	0.7359 (0.0284)
	0.1	0.1043 (0.0016)	0.1245 (0.0016)	0.0867 (0.0010)	0.0828 (0.0012)	0.1101 (0.0019)	0.1446 (0.0014)	0.1593 (0.0023)	0.0763 (0.0017)
20	2.0	1.8843 (0.2472)	2.2533 (0.2582)	1.7110 (0.2108)	1.8325 (0.2206)	2.1247 (0.2779)	1.7596 (0.2371)	2.4060 (0.3081)	2.1559 (0.2646)
	1.6	1.6178 (0.1664)	2.2085 (0.1670)	1.5039 (0.1343)	2.1023 (0.1434)	1.2285 (0.1812)	1.3780 (0.1503)	1.8694 (0.2008)	1.3859 (0.1652)
	0.8	0.5020 (0.0416)	0.9822 (0.0418)	0.5904 (0.0343)	0.8085 (0.0357)	0.6603 (0.0447)	0.8974 (0.0384)	0.7566 (0.0502)	1.0821 (0.0419)
	0.4	0.3258 (0.0100)	0.3320 (0.0104)	0.3291 (0.0085)	0.5556 (0.0087)	0.3862 (0.0112)	0.4160 (0.0096)	0.3345 (0.0124)	0.5044 (0.0104)
	0.1	0.1435 (0.0006)	0.0863 (0.0006)	0.0786 (0.0005)	0.0786 (0.0005)	0.0816 (0.0007)	0.0889 (0.0005)	0.0896 (0.0007)	0.1218 (0.0006)
30	2.0	1.8396 (0.1577)	2.0400 (0.1521)	2.4048 (0.1372)	2.3747 (0.1426)	2.6448 (0.1692)	1.3696 (0.1466)	1.7998 (0.1773)	2.3920 (0.1629)
	1.6	1.8986 (0.1009)	1.4451 (0.0985)	1.4127 (0.0900)	1.6934 (0.0926)	1.3902 (0.1091)	1.2850 (0.0943)	1.4352 (0.1186)	1.6527 (0.1065)
	0.8	0.6299 (0.0247)	0.9184 (0.0245)	0.7750 (0.0223)	0.8742 (0.0225)	0.9525 (0.0270)	1.0401 (0.0236)	0.6769 (0.0295)	0.6966 (0.0263)
	0.4	0.3403 (0.0063)	0.5601 (0.0063)	0.4575 (0.0055)	0.4343 (0.0057)	0.3231 (0.0065)	0.5006 (0.0059)	0.3467 (0.0072)	0.4466 (0.0064)
	0.1	0.1209 (0.0004)	0.1498 (0.0003)	0.0615 (0.0003)	0.0845 (0.0003)	0.1209 (0.0004)	0.1095 (0.0003)	0.0893 (0.0004)	0.0842 (0.0003)
40	2.0	1.5630 (0.1102)	1.8512 (0.1168)	1.8934 (0.1028)	2.5494 (0.1220)	1.9268 (0.1164)	2.0709 (0.1088)	1.7005 (0.12600	2.4434 (0.1114)
	1.6	1.3763 (0.0722)	1.4977 (0.0722)	1.5760 (0.0673)	1.4523 (0.0675)	1.9244 (0.0759)	1.4079 (0.0683)	1.4985 (0.08470	1.4310 (0.0717)
	0.8	0.9596 (0.0176)	0.6956 (0.0175)	0.7918 (0.0167)	0.7188 (0.0167)	0.7112 (0.0194)	0.7672 (0.0173)	0.7531 (0.0204)	0.7953 (0.0177)
	0.4	0.3641 (0.0044)	0.3584 (0.0043)	0.3527 (0.0040)	0.3992 (0.0041)	0.4714 (0.0047)	0.5147 (0.0043)	0.3648 (0.0051)	0.4628 (0.0046)
	0.1	0.0934 (0.0002)	0.1379 (0.0002)	0.0988 (0.0002)	0.1073 (0.0002)	0.0788 (0.0003)	0.0856 (0.0002)	0.1085 (0.0003)	0.0897 (0.0002)
50	2.0	1.8343 (0.0879)	1.7255 (0.0898)	2.1621 (0.0805)	1.8748 (0.0832)	1.9274 (0.0950)	2.4183 (0.0848)	2.0953 (0.0959)	2.1127 (0.0883)
	1.6	1.4348 (0.0575)	2.1334 (0.0547)	1.4778 (0.0524)	1.5521 (0.0542)	1.5047 (0.0585)	1.4641 (0.0545)	1.2454 (0.0635)	1.4796 (0.0570)
	0.8	0.8784 (0.0144)	0.7778 (0.0142)	0.8020 (0.0132)	0.7146 (0.0136)	1.0945 (0.0150)	0.6772 (0.0136)	0.8642 (0.0157)	0.8713 (0.0141)
	0.4	0.4168 (0.0034)	0.5448 (0.0036)	0.4865 (0.0033)	0.4095 (0.0033)	0.3582 (0.0037)	0.3660 (0.0033)	0.6553 (0.0038)	0.3436 (0.0035)
	0.1	0.0944 (0.0002)	0.0898 (0.0002)	0.1081 (0.0001)	0.0987 (0.0002)	0.1033 (0.0002)	0.1239 (0.0002)	0.1075 (0.0002)	0.0985 (0.0002)

From Table 3, it has been noticed that the MSE of DLF is quite high for such a tiny sample size but with increasing sample size, they drop drastically and become more comparable to other estimators. Among Bayes estimator, QLF gives the lower MSE's value, when the sample size is less but they're almost equal when compared to a large sample.

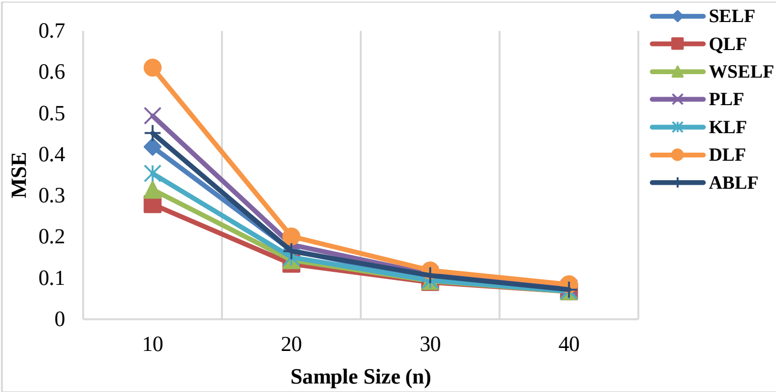

Figure 6: Graph of MSEs of Different Estimators of φ of Gompertz Lomax Distribution where $\alpha = 1$, $\beta = 1$, $\gamma = 1$, $\varphi = 2$ and $c = 0.1$

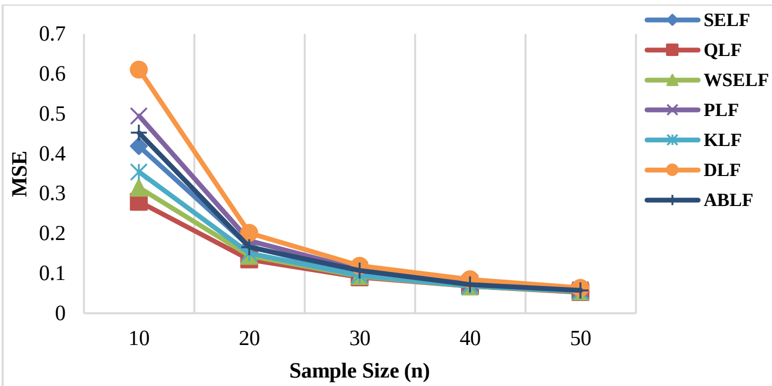

Figure 7: Graph of MSEs of Different Estimators of φ of Gompertz Lomax Distribution where $\alpha = 1$, $\beta = 1$, $\gamma = 1$, $\varphi = 1.6$ and $c = 0.1$

5. COMMENTS AND CONCLUSION

This study developed Bayesian estimators for the shape parameter of the GoLom Distribution using gamma, exponential, and Jeffreys's prior distributions with seven loss functions. The priors and loss functions are used to build and simplify this parameter's posterior distributions and Bayes estimators. For various parameter values and sample sizes, the performance of these estimators is evaluated based on their mean square errors using the inverse transformation method of Monte Carlo Simulations. The simulation and comparison results demonstrate that using the DLF and QLF generates estimators with reduced MSEs under the prior distributions (gamma and Jeffreys's priors). Specifically, regardless of the parameter values and sample sizes employed, the Bayesian Method with WSELF under exponential prior provides superior shape parameter estimators than estimators using DLF and QLF under both gamma and Jeffreys's priors. This study also discovered that variation in the values of the other three distribution parameters does not affect the efficiency with which the estimators of the predicted shape parameter performed; however, because this study only addresses one of the GoLom's forms factors, it is proposed that future research should look at the remaining three distribution parameters because in practice. The findings of the simulation research for calculating the estimators of GoLom's shape parameter when the other scale and shape parameters are known are summarized and tabulated in the following tables, which contain the estimated values and MSEs observed. It has been found that the MSE estimators of shape parameters decrease as the shape parameter value decreases. It has also been discovered that the MSEs of various estimators of form parameter drop as sample size increases. Finally, for all parameter values, a noticeable drop in MSEs has been seen with increasing sample size. It has been concluded from the foregoing analysis and graphical depiction that, except for a few situations, Bayes estimators under DLF, WSELF, and QLF outperform other estimators in the research. Based on previous knowledge, the Bayes estimator is used to determine posterior probability. In practice, the Bayes estimator may be used to anticipate insurance payouts.

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APPLICATION OF THE SIA-NAKAGAMI-RATIO DISTRIBUTION TO COVID-19 DATA FROM BRAZIL

Nowara Farooq[1], Saleha Naghmi Habibullah[2] and **Fatima Jamil[3]**
Kinnaird College for Women, Lahore, Pakistan
Email: [1]nowara57222@gmail.com
[2]saleha.habibullah@kinnaird.edu.pk
[3]fatima.jamil@kinnaird.edu.pk

ABSTRACT

Self-inverse distributions have the desirable property that they provide a mechanism for developing 'SIA estimators' which are more efficient than the widely used moment estimators. The SIA-Nakagami-Ratio distribution is a newly developed generalization of the SIU-Nakagami-Ratio distribution which possesses the property of self-inversion at unity. Whereas basic properties of the newly developed SIA distribution are given in a 2021 paper, in this work, we obtain some additional properties of the distribution including the mean residual life function, the Fisher information and the Shannon entroopy. As well, we demonstrate the applicability of the SIA-Nakagami-Ratio density function to real-world data by applying it to two COVID-19-related datasets from Brazil.

1. INTRODUCTION

The distributions of non-negative continuous random variables have a substantial role in distribution theory because of their relevance in domains such as reliability and survival analysis. In general, such distributions are unimodal and positively skewed, and may be used for modeling life data as well as other types of duration data. Farooq et al. (2021) generalized the SIU-Nakagami-Ratio distribution given by Jamil & Habibullah (2013) in order to obtain the SIA-Nakagami-Ratio probability model which has two parameters, making this new density function more adaptable to data as compared with the original SIU distribution. Farooq et al. (2021) demonstrated the self-inversion of the SIA density function and derived the essential characteristics of the distribution, including its shape, moments, quantile function, hazard rate and survival function.

In this research-work, we derive the mathematical expressions of the Mean Residual Life Function, the Fisher information and the Shannon entropy of the SIA-Nakagami-Ratio distribution and demonstrate the usefulness of this distribution for purposes of modeling real-life data by applying it to two datasets from Brazil pertaining to length of stay in hospital from admission to death using (i) the moment estimator and (ii) the SIA estimator of the distribution mean.

2. LITERATURE REVIEW

Habibullah and Ahmed (2006) generated a class of distributions that are closed under inversion, and developed some statistical properties of this class of probability distributions. Jones (2008) investigated various properties, analogies, comparisons and consequences of the variable having the same distribution as its reciprocal or as ordinary symmetry of the distribution of the logged random variable. Habibullah and Saunders (2011) introduced the concept of self-inversion at beta and adopted the nomenclature ‘self-inverse at unity’ for distributions that are invariant under the reciprocal transformation. Jamil & Habibullah (2013) derived the SIU-Nakagami-Ratio distribution by taking the ratio of the Nakagami random variable with itself. Fatima and Habibullah (2013) proposed a modification to the formula of the sample mean for the class of SIU distributions and showed that the modified mean is more efficient in estimating the distribution mean than the simple arithmetic mean of the sample data. Habibullah and Jamil (2014) focused on the self-inversion-based estimator of m and conducted a simulation study based on 1000 samples of various sizes drawn from the SIU-Nakagami-Ratio distribution, which showed the visible reduction in the variance. Habibullah and Fatima (2015) proposed a modification to the formula of the sample mean on the basis of the more general case ‘self-inversion at A’, where A can be any arbitrary real number. Habibullah (2017) asserted that log-symmetric distributions are essentially SIA and by applying the power-transformation, obtained the class of “SIA Log-Symmetric Power distributions”. The SIA property provides the capability of developing SIA-estimators that are more efficient than the ordinary ‘method of moments-type’ estimators.

Habibullah and Xavier (2018) considered a data-set pertaining to remission times of bladder cancer patients and applied one of the SIA log-symmetric distributions to this data-set. The results provide the better fit, in recognition of the more accurate estimation of probabilities of early remission for cancer-care providers. Habibullah and Xavier (2019) applied SIA log-symmetric distributions to a data-set taken from the literature for which the chi square test of goodness of fit testifies to the usefulness of the SIA-methodology. Farooq et al. (2021) generalized the SIU-Nakagami-Ratio distribution in order to obtain the SIA-Nakagami Ratio distribution which has two parameters and is more flexible than the original distribution. Basic properties including the shape of the distribution, moments, moment generating function, quantile function, measures of central tendency, dispersion, skewness and kurtosis were presented. Hazard function and survival function were derived. It was asserted by the authors that the positively skewed shape of the distribution indicates its applicability to real-life data.

As far as modeling of Covid-19 data is concerned, Hawryluk et al. (2020) have established epidemiological distributions for patients hospitalized with COVID-19 using a large dataset (N =21,000 – 157,000) from the Brazilian database. A joint Bayesian subnational model with partial pooling is used and strong evidence was found in favour of different probability density functions. The gamma distribution fitted well for onset-to-death and the generalized lognormal for onset-to-hospital-admission. The results of this study showed considerable geographical variation, and gave the first estimate of these epidemiological distributions in a low and middle-income setting. COVID-19 outcome timings were correlated with poverty, deprivation and segregation levels. Whereas mean age, wealth and urbanicity showed weaker correlation at the subnational level. Castro et al. (2021) used daily data of reported cases and deaths and observed high and unequal infection and mortality burden in case of an overall implementing prompt failure and equitable responses failure in the context of diversity in the spread of disease. Faria et al. (2021) investigated faster molecular evolution by using two categories of dynamical model that integrates genomic and mortality data and resulted that increased transmissibility and/or immune evasion, is critical to accelerate pandemic responsiveness.

3. MEAN RESIDUAL LIFE FUNCTION OF THE SIA-NAKAGAMI-RATIO DISTRIBUTION

The SIA Nakagami Ratio distribution is given by

$$g(t)=\frac{2\mathrm{A}^{2\mathrm{m}}}{\mathrm{B}(m,m)}t^{2m-1}\left(t^2+\mathrm{A}^2\right)^{-2m},\ 0<t<\infty, m>0, \mathrm{A}>0 \tag{3.1}$$

where t is a non-negative continuous random variable.

The survival function of the SIA-Nakagami-Ratio distribution is given by

$$\bar{\mathrm{G}}(t)=\frac{\mathrm{A}^{2\mathrm{m}}t^{-2m}\,{}_2F_1\left(m,2m,1+m,-\frac{\mathrm{A}^2}{t^2}\right)}{m\mathrm{B}(m,m)}$$

As such, the MRLF of the density is:

$$\mu(t)=\frac{1}{\bar{G}(t)}\int_t^\infty \bar{G}(t)\,dt=\frac{1}{\dfrac{\mathrm{A}^{2\mathrm{m}}t^{-2m}\,{}_2F_1\left(m,2m,1+m,-\frac{\mathrm{A}^2}{t^2}\right)}{m\mathrm{B}(m,m)}}\int_x^\infty \frac{\mathrm{A}^{2\mathrm{m}}t^{-2m}\,{}_2F_1\left(m,2m,1+m,-\frac{\mathrm{A}^2}{t^2}\right)}{m\mathrm{B}(m,m)}dt$$

$$=t\left(-1+\frac{2m\,{}_2F_1\left(-\frac{1}{2}+m,2m,\frac{1}{2}+m,-\frac{\mathrm{A}^2}{t^2}\right)}{(-1+2m)\,{}_2F_1\left(m,2m,1+m,-\frac{\mathrm{A}^2}{t^2}\right)}\right)$$

Figures 3.1(a) and 3.1(b) shows the MRLF of the density for two different values of the parameter m and five different values of the parameter A.

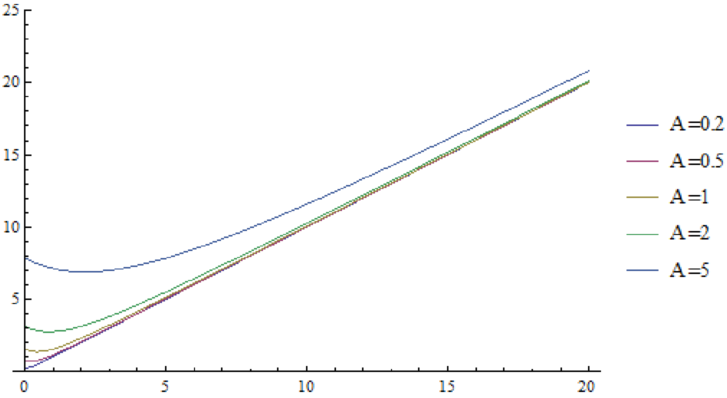

Figure 3.1(a): MRLF of the Density for *m*=1 where Parameter A has Multiple Values

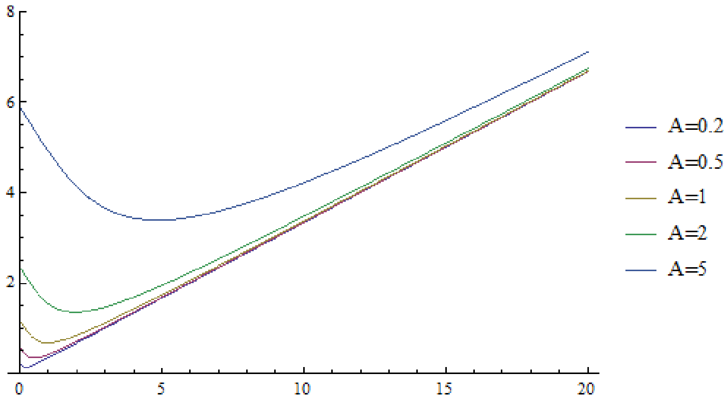

Figure 3.1(b): MRLF of the density for *m*=2 where Parameter A has Multiple Values

4. FISHER INFORMATION

Fisher Information is the amount of information conveyed by an observable random variable x regarding an unknown parameter on which the probability of x is based.

After taking the natural logarithm and the derivative with respect to its parameters, we obtain

$$\mathrm{I}(\mathrm{A},m)=-E\begin{bmatrix}\dfrac{\partial^2 \ln g(x)}{\partial^2 \mathrm{A}} & \dfrac{\partial \ln g(x)}{\partial \mathrm{A}m}\\ \dfrac{\partial \ln g(x)}{\partial m\mathrm{A}} & \dfrac{\partial^2 \ln g(x)}{\partial^2 m}\end{bmatrix}=\begin{bmatrix}\dfrac{4m^2}{\mathrm{A}^2(1+2m)} & 0\\ 0 & 2\psi^{(1)}(m)-4\psi^{(1)}(2m)\end{bmatrix}.$$

5. SHANNON ENTROPY

The average information in a communication was encapsulated by Shannon entropy. It is arguably the most sensitive metric for evaluating system diversity.

After integrating and simplifying, we obtain

$$E(-\ln g(x)) = -\ln\left(\frac{2\mathrm{A}^{2m}}{\mathrm{B}(m,m)}\right) - \frac{2\ln(\mathrm{A})}{\Gamma(2m)} + 2m(2\ln \mathrm{A} - \psi^{0}(m) + \psi^{0}(2m))$$

Here $\psi^{(0)}(m)$ & $\psi^{(0)}(2m)$ are digamma or polygamma functions.

6. SIMULATION STUDY

In this section, we use the self-inversion based and simple mean estimators proposed by Fatima & Habibullah (2013) in order to find mean, standard deviation, coefficient of variation by conducting simulation study.

The simulation study is carried out through an R program by using the quantile function of the SIA-Nakagami-Ratio distribution for m=1 turns out to be

$$x_p = \mathrm{A}\sqrt{\frac{p}{(1-p)}}$$

This simulation study is based on 75000 samples of various sizes ‘n’ drawn from the SIA-Nakagami-Ratio distribution for different of the parameter A.

Table 6.1
Mean, S.D. and C.V. with ‘SIA-Mean’ and Simple Mean for A=1

n	Mean of Means	S.D. of Means	C.V. of Means	Mean of SIA-Means	S.D. of SIA-Means	C.V. of SIA-Means
25	1.573167	0.7191257	45.71197%	1.570978	0.4639186	29.53057%
50	1.569925	0.4757178	30.30195%	1.570648	0.3314398	21.10211%
100	1.569656	0.3852488	24.54351%	1.570081	0.2854481	18.18047%

Table 6.2
Mean, S.D. and C.V. with ‘SIA-Mean’ and Simple Mean for A=2

n	Mean of Means	S.D. of Means	C.V. of Means	Mean of SIA-Means	S.D. of SIA-Means	C.V. of SIA-Means
25	3.143481	1.326251	42.19051	3.141347	0.9584122	30.5096
50	3.141282	1.047467	33.34522	3.14312	0.8312605	26.44699
100	3.141175	0.7021818	22.35412	3.140664	0.5222589	16.62893

Table 6.3
Mean, S.D. and C.V. with 'SIA-Mean' and Simple Mean for A=3

n	Mean of Means	S.D. of Means	C.V. of Means	Mean of SIA-Means	S.D. of SIA-Means	C.V. of SIA-Means
25	4.725066	2.245271	47.5183	4.712921	1.410393	29.92609
50	4.716525	1.616297	34.26881	4.71377	1.045216	22.17368
100	4.715556	1.196763	25.37903	4.711487	0.8616432	18.28814

Table 6.4
Mean, S.D. and C.V. with 'SIA-Mean' and Simple Mean for A=4

n	Mean of Means	S.D. of Means	C.V. of Means	Mean of SIA-Means	S.D. of SIA-Means	C.V. of SIA-Means
25	6.307473	3.478028	55.14139	6.288904	2.178802	34.64517
50	6.284661	1.915313	30.47599	6.281495	1.426161	22.70417
100	6.278833	1.487715	23.69414	6.280824	0.9915406	15.78679

Table 6.5
Mean, S.D. and C.V. with 'SIA-Mean' and Simple Mean for A=5

n	Mean of Means	S.D. of Means	C.V. of Means	Mean of SIA-Means	S.D. of SIA-Means	C.V. of SIA-Means
25	7.867621	3.642429	46.29644	7.862776	2.327301	29.59897
50	7.865564	2.657895	33.79153	7.855613	1.720136	21.8969
100	7.847217	1.852329	23.60492	7.848409	1.170152	14.90942

The above Tables 6.1 to 6.5 shows the visible reduction in the coefficient of variation of the SIA-means as compared with the ordinary means for different sample sizes.

7. APPLICATION TO COVID-19 DATASETS

In this section, we consider COVID-19 patient's data from Brazilian hospital presented by Hawryluk et al. (2020) in which hospital-admission-to-death (length of stay) has been analyze. In the dataset, variable x representing hospital-admission-to-death (length of stay) is analyzed for two states of Brazil, Acre (AC) and Mato Grosso do Sul (MS).

We employ Kolmogorov-Smirnov (KS) goodness of fit test to each of the two datasets by using both the simple mean and the self-inversion-based estimator of the distribution mean.

The first dataset is related to the Acre (AC) state of Brazil. The length of stay of patients who stay in hospital from admission to death has been analyzed. The total number of patients are 108. The histogram of the dataset is shown in Figure 7.1 which clearly indicates that the data are positively skewed.

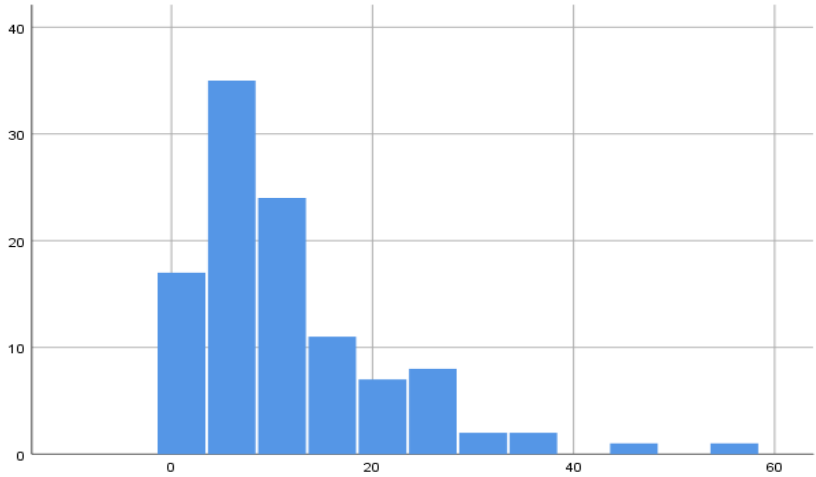

Figure 7.1: Histogram of COVID-19 Hospital-Admission-to-Death (Length of Stay) Patients Data in Acre (AC) State of Brazil

Table 7.1 describes the results of K-S Goodness of fit test for simple mean and by SIA Estimator of μ. In both cases, the value of the K-S Statistic comes out to be less than the critical value at 5% level of significance. We can also observe from the table that the value of K-S Statistic for the SIA Estimator of μ is smaller than the value in the case of the simple mean which provides clear indication that the SIA-estimator of the distribution mean is providing a better fit than the simple mean.

Table 7.1
Result of K-S Test of Goodness of Fit by Simple Mean and SIA-Estimator of μ

Critical Value	0.130866
D in the Case of Simple Mean	0.109284
D in the Case of SIA-Estimator of Distribution Mean	0.074907

The second dataset is related to the Mato Grosso do Sul (MS) state of Brazil. The COVID-19 patients' time interval (length of stay in hospital, admission to death) has been analyzed. The data consist of 116 patients. The histogram of the dataset is shown in Figure 7.2 which clearly indicates that the data are positively skewed.

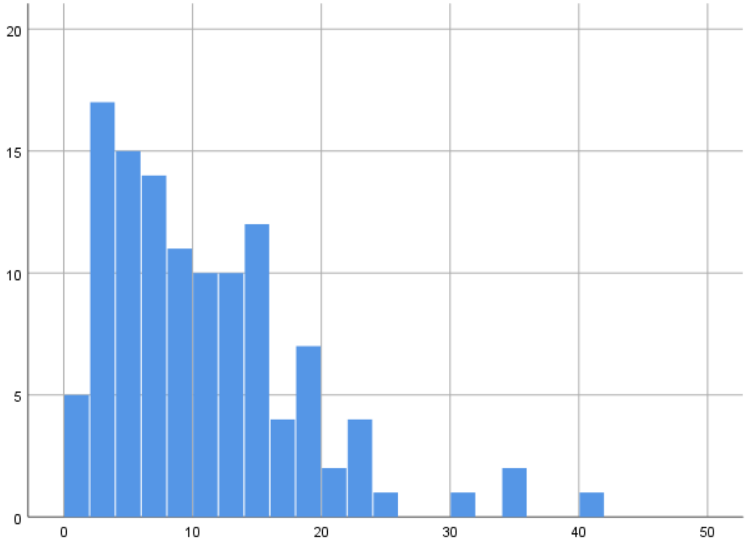

Figure 10.2: Histogram of COVID-19 Hospital-Admission-to-Death (length of stay) Patients Data in Mato Grosso do Sul (MS) state of Brazil

Table 7.2 shows the results of K-S Goodness of fit test in the case of the simple mean and the SIA-estimator of μ. We observe from the table that the K-S statistic value for simple mean is greater than the critical value whereas in the case of the SIA-estimator, the K-S statistic value is insignificant at 5% level of significance. We can also observe from the table that the value of K-S statistics in the case of the SIA-estimator of μ is smaller than in the case of the simple mean which provides clear indication that the SIA-estimator of the distribution mean is providing a better fit than the simple mean.

Table 7.2
Result of K-S Test of Goodness of Fit by Simple Mean and SIA-Estimator of μ

Critical Value	0.1262728
D in the Case of Simple Mean	0.204475
D in the Case of SIA-Estimator of Distribution Mean	0.109003

8. COMMENTS AND CONCLUSION

In this study, we have taken up the SIA-Nakagami-Ratio distribution the shape of which is moderately positively skewed. The mean residual life function has been derived. The Fisher information and Shannon entropy are also derived.

The distribution has been applied on two datasets on the length of stay in hospital from admission to death of COVID-19 patients in two States of Brazil. The distribution has fitted the datasets well by using the method of moments. However, by employing the SIA-estimator of the mean for estimation of parameter *m*, the distribution has fitted better than in the case of utilization of the simple mean.

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GENERALIZED FORMS OF THE KULIB EXTENDED GAMMA AND THE BETA FUNCTIONS AND THEIR PROPERTIES

Maira Mobeen, Ghulam Farid, Zainab Javid and **Sonia Parveen**
Department of Mathematics and Statistics,
University of Agriculture, Faisalabad, Pakistan
Email: mairamobeen70@gmail.com
ghulamfarid@uaf.edu.pk
zainikhilji98@gmail.com
soniaparveen614@gmail.com

ABSTRACT

In this work, generalized forms of Kulib Gamma and Beta functions are proposed. The generalizations are based on a four parametric generalized form of the Wright function. Moreover, the Mellin transform and some properties like functional relation, integral transform and summation formulae of the generalized Gamma and Beta functions are discussed. Moreover, the relation between generalized forms of Kulib Gamma and Beta functions is also established.

Mathematics Subject Classification: 11S80, 33B15, 97I50.

KEYWORDS

Gamma function, Beta function, Wright function, Mellin transform.

1. INTRODUCTION

The integral forms of the classical Gamma and Beta functions, respectively are

$$\Gamma(r) = \int_0^\infty t^{r-1} e^{-t} dt, \, r > 0$$

and
$$B(r,s) = \int_0^1 t^{r-1}(1-t)^{s-1} dt, \; r,s > 0. \tag{1}$$

Due to extensive applications of the Gamma and Beta functions in various fields of sciences and engineering, a number of extensions and generalized forms of the Gamma and Beta functions have been introduced and are available in the literature. For further study of these topics we refer, Silverman (1972), Davis (1959), Andrews *et al*. (1999), Temme (1996) and Artin (2015).

E.M. Wright presented Wright function, given in the following form

$$\varphi_\lambda(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(\lambda n+1)n!}, \; \lambda \in C.$$

The four parametric generalized form of the Wright function is defined as

$$\varphi_{\lambda,\mu}^{\delta,\kappa}(z)=\sum_{n=0}^{\infty}\frac{(\delta)_n}{(\kappa)_n\Gamma(\lambda n+\mu)}\frac{z^n}{n!},\ \lambda,\mu,\delta,\kappa\in C.$$

Kulib *et al*. (2020) introduced extension of Gamma and Beta functions, as

$${}^{w}\Gamma_{p}^{(\lambda,\mu,\delta,\kappa)}(r)=\int_0^\infty t^{r-1}W_{\lambda,\mu}^{\delta,\kappa}\left(-t-\frac{p}{t}\right)dt$$

and $${}^{w}B_{p}^{(\lambda,\mu,\delta,\kappa)}(r,s)=\int_0^1 t^{r-1}(1-t)^{s-1}W_{\lambda,\mu}^{\delta,\kappa}\left(-\frac{p}{t(1-t)}\right)dt$$

respectively, in which $p\geq 0; r,s>0; \lambda,\mu,\delta,\kappa\in C$.

2. GENERALIZATION OF KULIB GAMMA AND BETA FUNCTIONS

We define generalized forms of Kulib Gamma and Beta functions as

$${}^{\varphi}\Gamma_{p,n}^{(\lambda,\mu,\delta,\kappa)}(r)=\int_0^\infty t^{r-1}\varphi_{\lambda,\mu}^{\delta,\kappa}\left(-t-\frac{p}{t}\right)^n dt \tag{2}$$

and $${}^{\varphi}B_{p,q;k,n}^{(\lambda,\mu,\delta,\kappa)}(r,s)=\int_0^1 t^{r-1}(1-t)^{s-1}\varphi_{\lambda,\mu}^{\delta,\kappa}\left(\frac{-p}{t^k}\right)^n\varphi_{\lambda,\mu}^{\delta,\kappa}\left(\frac{-q}{(1-t)^k}\right)^n dt, \tag{3}$$

respectively, in which $p,q\geq 0; k,n,r,s>0; \lambda,\mu,\delta,\kappa\in C.$

2. MAIN RESULTS

From 0, it is easy to see that:

i) For $p=0$;

$${}^{\varphi}\Gamma_{0,n}^{\lambda,\mu,\delta,\kappa}(r)=\int_0^\infty t^{r-1}\varphi_{\lambda,\mu}^{\delta,\kappa}(-t)^n dt={}^{\varphi}\Gamma_{n}^{\lambda,\mu,\delta,\kappa}(r). \tag{4}$$

ii) For $n=1$; ${}^{\varphi}\Gamma_{p,1}^{\lambda,\mu,\delta,\kappa}(r)={}^{\varphi}\Gamma_{p}^{\lambda,\mu,\delta,\kappa}(r).$

From 0, it is easy to see that

i) For $\lambda=\mu=\delta=\kappa=1$ and $p=q$; ${}^{\varphi}B_{p,p;k,n}^{(1,1,1,1)}(r,s)={}^{\varphi}B_{p,k,n}(r,s).$

ii) If $n=k=1$ and $\lambda=\mu=\delta=\kappa=1$, ${}^{\varphi}B_{p,q;1,1}^{(1,1,1,1)}(r,s)={}^{\varphi}B_{p,q}(r,s).$

Theorem:

Let $p,q\geq 0; k,n,r,s>0$ and $\lambda,\mu,\delta,\kappa\in C$. Then, the product of two generalized form of Kulib Gamma functions satisfies the relation

$$^{\varphi}\Gamma_n^{(\lambda,\mu,\delta,\kappa)}(u)\,^{\varphi}\Gamma_n^{(\lambda,\mu,\delta,\kappa)}(v)=\frac{1}{B\left(r+ku,s+kv\right)}\int_0^\infty\int_0^\infty p^{u-1}q^{v-1}\,^{\varphi}B_{p,q;k,n}^{(\lambda,\mu,\delta,\kappa)}(r,s)dpdq. \quad (5)$$

Proof:

Let

$$I={}^{\varphi}\Gamma_n^{(\lambda,\mu,\delta,\kappa)}(u)\,^{\varphi}\Gamma_n^{(\lambda,\mu,\delta,\kappa)}(v)\,. \quad (6)$$

Then, by use of **Error! Reference source not found.**, it follows that

$$I\;=\frac{B(r+ku,s+vk)}{B(r+ku,s+vk)}\int_0^\infty\int_0^\infty w^{u-1}z^{v-1}\varphi_{\lambda,\mu}^{\delta,\kappa}(-w)^n\,\varphi_{\lambda,\mu}^{\delta,\kappa}(-z)^n\,dwdz.$$

Using 0, we obtain

$$I=\frac{1}{B(r+ku,s+kv)}\int_0^1 t^{r+uk-1}(1-t)^{s+vk-1}dt\int_0^\infty\int_0^\infty w^{u-1}z^{v-1}\varphi_{\lambda,\mu}^{\delta,\kappa}(-w)^n\,\varphi_{\lambda,\mu}^{\delta,\kappa}(-z)^n\,dwdz.$$

For $w=\dfrac{p}{t^k}$ and $z=\dfrac{q}{(1-t)^k}$; the above relation becomes

$$I=\frac{1}{B(r+ku,s+kv)}\int_0^1 t^{r-1}(1-t)^{s-1}g(t)\,dt,$$

where $g(t)=\left\{\int_0^\infty\int_0^\infty t^{ku}(1-t)^{kv}\left(\frac{p}{t^k}\right)^{u-1}\left(\frac{q}{(1-t)^k}\right)^{v-1}\varphi_{\lambda,\mu}^{\delta,\kappa}\left(\frac{-p}{t^k}\right)^n\varphi_{\lambda,\mu}^{\delta,\kappa}\left(\frac{-q}{(1-t)^k}\right)^n\frac{dp}{t^k}\frac{dq}{(1-t)^k}\right\}dt$

by use of 0, we get

$$I=\frac{1}{B(r+ku,s+kv)}\int_0^\infty\int_0^\infty p^{u-1}q^{v-1}\,^{\varphi}B_{p,q;k,n}^{(\lambda,\mu,\delta,\kappa)}(r,s)dpdq. \quad (7)$$

Comparing 0 and 0, we have the desired result 0.

Theorem (**Mellin Transform)**

Suppose that $p,q\geq 0;k,n,r,s>0$ and $\lambda,\mu,\delta,\kappa\in C$. Then, the Mellin transform of generalized form of Kulib Beta function is given by the relation

$$M\left\{^{\varphi}B_{p,q;k,n}^{(\lambda,\mu,\delta,\kappa)}(r,s);p\to x,q\to y\right\}=B(r+kx,s+ky)\,^{\varphi}\Gamma_n^{(\lambda,\mu,\delta,\kappa)}(x)\,^{\varphi}\Gamma_n^{(\lambda,\mu,\delta,\kappa)}\left(y\right). \quad (8)$$

Proof:

Suppose

$$I=M\left\{^{\varphi}B_{p,q;k,n}^{(\lambda,\mu,\delta,\kappa)}(r,s);p\to x,q\to y\right\}. \quad (9)$$

by use of 0, we obtain

$$I=\int_0^1 t^{r-1}(1-t)^{s-1}\left\{\int_0^\infty p^{x-1}\varphi_{\lambda,\mu}^{\delta,\kappa}\left(\frac{-p}{t^k}\right)^n dp\int_0^\infty q^{y-1}\varphi_{\lambda,\mu}^{\delta,\kappa}\left(\frac{-q}{(1-t)^k}\right)^n dq\right\}dt. \tag{10}$$

For $u=\dfrac{p}{t^k}$ and $v=\dfrac{q}{(1-t)^k}$; 0 leads to the relation

$$I=\int_0^1 t^{r-1}(1-t)^{s-1}\left\{\int_0^\infty (ut^k)^{x-1}\varphi_{\lambda,\mu}^{\delta,\kappa}(-u)^n t^k du\int_0^\infty \left((1-t)^k v\right)^{y-1}\varphi_{\lambda,\mu}^{\delta,\kappa}(-v)^n(1-t)^k dv\right\}dt.$$

After simplification, we have

$$I=\int_0^1 t^{r+kx-1}(1-t)^{s+ky-1}\left\{\int_0^\infty u^{x-1}\varphi_{\lambda,\mu}^{\delta,\kappa}(-u)^n du\int_0^\infty v^{y-1}\varphi_{\lambda,\mu}^{\delta,\kappa}(-v)^n dv\right\}dt.$$

By using 0 and **Error! Reference source not found.**, we obtain

$$I=B(r+kx,s+ky)\,{}^{\varphi}\Gamma_n{}^{(\lambda,\mu,\delta,\kappa)}(x)\,{}^{\varphi}\Gamma_n{}^{(\lambda,\mu,\delta,\kappa)}(y). \tag{11}$$

The relations 0 and 0 lead to the desired formula.

Corollary:

For $p,q\geq 0; k,n,r,s>0; \lambda,\mu,\delta,\kappa\in C$ and $c_1,c_2>0;$ the Mellin inverse transformation on the generalized form of Kulib Beta function is given by the following relation

$${}^{\varphi}B_{p,q;n;k}^{(\lambda,\mu,\delta,\kappa)}(r,s)=\frac{1}{(2\pi i)^2}\int_{c_1-i\infty}^{c_1+i\infty}\int_{c_2-i\infty}^{c_2+i\infty}B(r+kx,s+ky)\,{}^{\varphi}\Gamma_n{}^{(\lambda,\mu,\delta,\kappa)}(x)\,{}^{\varphi}\Gamma_n{}^{(\lambda,\mu,\delta,\kappa)}\left(y\right).$$

Theorem (Functional Relation):

Let $p,q\geq 0; k,n,r,s>0$ and $\lambda,\mu,\delta,\kappa\in C$.Then the following functional relation, for the generalized form of Kulib Beta function, holds

$${}^{\varphi}B_{p,q;k,n}^{(\lambda,\mu,\delta,\kappa)}(r+1,s)+{}^{\varphi}B_{p,q;k,n}^{(\lambda,\mu,\delta,\kappa)}(r,s+1)={}^{\varphi}B_{p,q;k,n}^{(\lambda,\mu,\delta,\kappa)}(r,s). \tag{12}$$

Proof:

Let

$$I={}^{\varphi}B_{p,q;k,n}^{(\lambda,\mu,\delta,\kappa)}(r+1,s)+{}^{\varphi}B_{p,q;k,n}^{(\lambda,\mu,\delta,\kappa)}(r,s+1). \tag{13}$$

By using 0, we get

$$I=\int_0^1 t^r(1-t)^{s-1}\varphi_{\lambda,\mu}^{\delta,\kappa}\left(\frac{-p}{t^k}\right)^n\varphi_{\lambda,\mu}^{\delta,\kappa}\left(\frac{-q}{(1-t)^k}\right)^n dt+\int_0^1 t^{r-1}(1-t)^s\varphi_{\lambda,\mu}^{\delta,\kappa}\left(\frac{-p}{t^k}\right)^n\varphi_{\lambda,\mu}^{\delta,\kappa}\left(\frac{-q}{(1-t)^k}\right)^n dt.$$

Then, after simplification, we obtain

$$I=\int_0^1 t^{r-1}(1-t)^{s-1}\varphi_{\lambda,\mu}^{\delta,\kappa}\left(\frac{-p}{t^k}\right)^n\varphi_{\lambda,\mu}^{\delta,\kappa}\left(\frac{-q}{(1-t)^k}\right)^n dt.$$

Using 0, it follows that

$$I = {}^{\varphi}B_{p,q;k,n}^{(\lambda,\mu,\delta,\kappa)}(r,s). \tag{14}$$

The relations 0 and 0 lead the required result 0.

Theorem:

Let $p, q \geq 0; k, n, r, s > 0$ and $\lambda, \mu, \delta, \kappa \in C$. Then, for the generalized form of Kulib Beta function, the following symmetric relation, functional relation and summation formula hold

i) ${}^{\varphi}B_{p,q;k,n}^{(\lambda,\mu,\delta,\kappa)}(r,s) = {}^{\varphi}B_{p,q;k,n}^{(\lambda,\mu,\delta,\kappa)}(s,r),$

ii) ${}^{\varphi}B_{p,q;k,n}^{(\lambda,\mu,\delta,\kappa)}(r+1,s) + {}^{\varphi}B_{p,q;k,n}^{(\lambda,\mu,\delta,\kappa)}(r,s+1) = {}^{\varphi}B_{p,q;k,n}^{(\lambda,\mu,\delta,\kappa)}(r,s)$

and

iii) $\sum_{i=0}^{j} \overset{j}{\underset{i}{C}} \; {}^{\varphi}B_{p,q;k,n}^{(\lambda,\mu,\delta,\kappa)}(r+i, s+j-i) = {}^{\varphi}B_{p,q;k,n}^{(\lambda,\mu,\delta,\kappa)}(r,s), \; j \in W.$

Proof:

By the use of 0, these relations can be easily verified.

Theorem (Integral Transform):

Let that $p, q \geq 0; k, n, r, s > 0$ and $\lambda, \mu, \delta, \kappa \in C$. Then, the generalized form of Kulib Beta function can be written in the following forms

$${}^{\varphi}B_{p,q;k,n}^{(\lambda,\mu,\delta,\kappa)}(r,s) = 2\int_0^{\frac{\pi}{2}} \cos^{2r-1}\theta \sin^{2s-1}\theta \, \varphi_{\lambda,\mu}^{\delta,\kappa}\left(\frac{-p}{\cos^{2k}\theta}\right)^n \varphi_{\lambda,\mu}^{\delta,\kappa}\left(\frac{-q}{\sin^{2k}\theta}\right)^n d\theta, \tag{15}$$

$${}^{\varphi}B_{p,q;k,n}^{(\lambda,\mu,\delta,\kappa)}(r,s) = \int_0^{\infty} \frac{v^{r-1}}{(1+v)^{r+s}} \varphi_{\lambda,\mu}^{\delta,\kappa}\left(\frac{-p(1+v)^k}{v^k}\right)^n \varphi_{\lambda,\mu}^{\delta,\kappa}\left(-q(1+v)^k\right)^n dv \tag{16}$$

and $${}^{\varphi}B_{p,q;k,n}^{(\lambda,\mu,\delta,\kappa)}(r,s) = (b-a)^{1-r-s}\int_a^b (u-a)^{r-1}(b-u)^{s-1} \varphi_{\lambda,\mu}^{\delta,\kappa}\left(\frac{-p(b-a)^k}{(u-a)^k}\right)^n \varphi_{\lambda,\mu}^{\delta,\kappa}\left(\frac{-q(b-a)^k}{(b-u)^k}\right)^n du. \tag{17}$$

Proof:

Putting $t = \cos^2\theta$ in 0, we obtain

$${}^{\varphi}B_{p,q;k,n}^{(\lambda,\mu,\delta,\kappa)}(r,s) = -2\int_{\frac{\pi}{2}}^{0} (\cos^2\theta)^{r-1}(1-\cos^2\theta)^{s-1} \varphi_{\lambda,\mu}^{\delta,\kappa}\left(\frac{-p}{(\cos^2\theta)^k}\right)^n \varphi_{\lambda,\mu}^{\delta,\kappa}\left(\frac{-q}{(1-\cos^2\theta)^k}\right)^n \cos\theta\sin\theta d\theta,$$

which, after simplification, becomes

$${}^{\varphi}B_{p,q;k,n}^{(\lambda,\mu,\delta,\kappa)}(r,s) = 2\int_0^{\frac{\pi}{2}} \cos^{2r-1}\theta \sin^{2s-1}\theta \, \varphi_{\lambda,\mu}^{\delta,\kappa}\left(\frac{-p}{\cos^{2k}\theta}\right)^n \varphi_{\lambda,\mu}^{\delta,\kappa}\left(\frac{-q}{\sin^{2k}\theta}\right)^n d\theta,$$

which is 0. Similarly, 0 and 0 can be proved by substituting $t=\frac{v}{1+v}$ and $t=\frac{u-a}{b-a}$, in 0, respectively.

Theorem:

Suppose that $p, q \geq 0; n, r > 0$ and $\lambda, \mu, \delta, \kappa \in C$. Then, the generalized form of Kulib Gamma function can be written in integral forms, as

$$ {}^{\varphi}\Gamma_n^{(\lambda,\mu,\delta,\kappa)}(r) = \int_0^{\frac{\pi}{2}} \frac{\sin^{r-1}\theta}{\cos^{r+1}\theta}\, \varphi_{\lambda,\mu}^{\delta,\kappa}\left(-\frac{\sin^2\theta + p\cos^2\theta}{\cos\theta\sin\theta}\right)^n d\theta, \tag{18} $$

$$ {}^{\varphi}\Gamma_n^{(\lambda,\mu,\delta,\kappa)}(r) = -\int_{-1}^{0} \frac{u^{r-1}}{(u+1)^{r+1}}\, \varphi_{\lambda,\mu}^{\delta,\kappa}\left(\frac{-u^2 - p(u+1)^2}{u(u+1)}\right)^n du \tag{19} $$

and $$ {}^{\varphi}\Gamma_n^{(\lambda,\mu,\delta,\kappa)}(r) = -\int_{0}^{1} \frac{u^{r-1}}{(u-1)^{r+1}}\, \varphi_{\lambda,\mu}^{\delta,\kappa}\left(\frac{-u^2 - p(u-1)^2}{u(u-1)}\right)^n du. \tag{20} $$

Proof:

For $t = \tan\theta$; 0 leads to the relation

$$ {}^{\varphi}\Gamma_n^{(\lambda,\mu,\delta,\kappa)}(r) = \int_0^{-1} \left(\frac{u}{u+1}\right)^{r-1} \varphi_{\lambda,\mu}^{\delta,\kappa}\left(-\frac{u}{u+1} - \frac{p(u+1)}{u}\right)^n \frac{1}{(u+1)^2}\, du. $$

After some simplification, we get the result 0. The relations 0 and 0 can be verified by putting $t=\frac{u}{u+1}$ and $t=\frac{u}{u-1}$, in 0, respectively.

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SOME GENERALIZED FORMULAE FOR ENCRYPTION AND DECRYPTION USING THE NATURAL TRANSFORM

Zainab Javid[1], **Ghulam Farid**[1], **Wania Javid**[2]
Sonia Parveen[1] and **Amina Firdous**[1]

[1] Department of Mathematics and Statistics
University of Agriculture, Faisalabad, Pakistan
Email: zainikhilji98@gmail.com
ghulamfarid@uaf.edu.pk
soniaparveen614@gmail.com
aminafirdous414@gmail.com

[2] Department of Physics, University of Agriculture
Faisalabad, Pakistan
Email: waniajaved99@gmail.com

ABSTRACT

In the proposed work, the Natural transform and the inverse Natural transform are applied on some functions in the form of some generalized power series. A single key is used for encrypting and decrypting the plaintext and the cipher text, respectively. Moreover, some formulae are established to obtain generalized results for cipher text, key and plaintext. Some new iterative formulae are also explored by applying the Natural transform successively. To convert plaintext and cipher text messages into mathematical form, numeric values from zero to sixty-seven are assigned to small alphabets, capital alphabets, six special characters and numbers from zero to nine. Finally, the work is elaborated with an example.

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KEYWORDS

Natural transform, symmetric cryptography, cryptology, cryptography, artificial intelligence, generalized functions

1. INTRODUCTION

Cryptology is a mathematical art to convert a readable message in an unreadable form using a key(s) and vice versa. The process of converting a readable message (plaintext) in an unreadable form (cipher text) is known as encryption while its converse process is named as decryption. If a single key is used for encryption and decryption, then it is known as symmetric cryptography whereas if different keys are generated for these processes, then it is known as asymmetric cryptography. Several mathematical techniques are engaged in these processes to get a cipher text.

Cryptography has been a topic of great concern since last century due to its extensive applications in all walks of life including artificial agencies, data computing, cloud computing, banking and almost in all related fields. A number of researchers have worked on this topic and have presented a number of techniques to secure data from any suspicious person. To study these techniques, we suggest Aumasson (2017), McAndrew (2016), Stanoyevitch (2010), Hiwareker (2013) and Khan and Khan (2008).

Hiwareker (2015) has presented a different iterative technique for cryptology in which the successive Laplace transformation of appropriate functions for encryption and decryption were applied. The implementation strategies were also illustrated. Alure (2017) has explored the applications of Laplace transformation in the field of cryptology. Gencoglu (2017) has given a general attack scene for performing security analysis of the Laplace transformation based cryptosystems that are symmetric by simple calculations of key. Mehmood et al. (2020) have employed Natural transform and its inverse for encryption and decryption respectively. In this work, generalized iterative formulae will be formulated by applying Natural transform and its inverse.

1.1 A standard result on the Natural Transform

By definition of Natural Transform, we know that

$$N\{g(t)\} = \int_0^\infty e^{-st} g(u,t)dt = R(u,s)$$

So,

$$\begin{aligned} N\{tg(t)\} &= \int_0^\infty e^{-st}(ut)g(u,t)dt \\ &= u\int_0^\infty g(u,t)te^{-st}dt \\ &= u\int_0^\infty g(u,t)(-\frac{d}{ds}e^{-st})dt \\ &= u(-\frac{d}{ds})R(u,s) \end{aligned}$$

Similarly, we can write

$$N\{t^n g(t)\} = u^n(-\frac{d}{ds})^n R(u,s)\ .$$

1.2 Encoding and Decoding Table

Plaintext will be converted into cipher text (and vice versa) by using the following encryption and decryption table:

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Q	R	S	T	U	V	W	X	Y	Z	a	b	c	d	e	f
16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v
32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47
w	x	y	z		0	1	2	3	4	5	6	7	8	9	!
48	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63
@	#	$	?												
64	65	66	67												

2. MAIN RESULTS

An example is presented to elaborate the proposed technique. We suppose ***"Look Back"*** is the plaintext that we want to convert into the cipher text. We convert plaintext into ciphertext by using section 1.2.

2.1 Encryption Process

L	*o*	*o*	*k*		*B*	*a*	*c*	*k*
11	*40*	*40*	*36*	*52*	*1*	*26*	*28*	*36*
M_0	M_1	M_2	M_3	M_4	M_5	M_6	M_7	M_8

Here $M_i = 0, \forall\, i \geq 9$.

Now, we consider the function in the form given below:

$$f(t) = t^a \sinh(bt) = \sum_{i=0}^{m} \frac{b^{2i+1} t^{2i+1+a}}{(2i+1)!};$$

If we put $(a,b) = (2,3)$ and multiply with the message, the above series takes the form;

$$\begin{aligned} Mf(t) = Mt^2 \sinh(3t) &= \sum_{i=0}^{8} M_i \frac{3^{2i+1} t^{2i+3}}{(2i+1)!} \\ &= 11\frac{3t^3}{1!} + 40\frac{3^3 t^5}{3!} + 40\frac{3^5 t^7}{5!} + 36\frac{3^7 t^9}{7!} + 52\frac{3^9 t^{11}}{9!} + 1\frac{3^{11} t^{13}}{11!} \\ &\quad + 26\frac{3^{13} t^{15}}{13!} + 28\frac{3^{15} t^{17}}{15!} + 36\frac{3^{17} t^{19}}{17!}. \end{aligned}$$

Now, we apply Natural transform and obtain the following values

$$N(Mf(t)) = N(Mt^2\sinh(3t)) = 33\frac{u^3 3!}{s^4} + 180\frac{u^4 5!}{s^6} + 81\frac{u^7 7!}{s^8} + 15.62\frac{u^9 9!}{s^{10}} + 2.82\frac{u^{11} 11!}{s^{12}} + 4.44\times10^{-3}\frac{u^{13} 13!}{s^{14}} + 6.66\times10^{-3}\frac{u^{15} 15!}{s^{16}} + 3.07\times10^{-4}\frac{u^{17} 17!}{s^{18}} + 1.31\times10^{-5}\frac{u^{19} 19!}{s^{20}}.$$

Particularly, we put $u = 2$, and get the following

$$=1584\frac{1}{s^4} + 691200\frac{1}{s^6} + 52254720\frac{1}{s^8} + 2902376448\frac{1}{s^{10}} + 2.30578\times10^{11}\frac{1}{s^{12}} + 2.26385\times10^{11}\frac{1}{s^{14}} + 2.85246\times10^{14}\frac{1}{s^{16}} + 1.43237\times10^{16}\frac{1}{s^{18}} + 8.33604\times10^{17}\frac{1}{s^{20}}.$$

If the above values are of the form of R_i ,

1584	R_0	691200	R_1
52254720	R_2	2902376448	R_3
2.30578×10^{11}	R_4	2.26385×10^{11}	R_5
2.85246×10^{14}	R_6	1.43237×10^{16}	R_7
8.33604×10^{17}	R_8		

Now, we take *mod* 68 of above values

20	48	52	40	12	60	52	36	24

Say above values of the form of M_i'

20	48	52	40	12	60	52	36	24
M_0'	M_1'	M_2'	M_3'	M_4'	M_5'	M_6'	M_7'	M_8'

By converting these numerical values into characters using section 1.2, we get cipher text i.e.

Uw oM8 kY

The ciphertext form M_i' of plaintext M_i can be generalized as

$$M_i' = [M_i b^{2i+1}(2i+1+1)\ldots(2i+1+a)u^{2i+1+a}] \bmod 68 = R_i - 68K_i ,$$

where $R_i = M_i b^{2i+1}(2i+1+1)\ldots(2i+1+a)u^{2i+1+a}$ and $K_i = \dfrac{R_i - M_i'}{68}$.

The cipher text form M_i'' of plaintext M_i' can be obtained simply by using the formula given below:

$$M_i''=[M_i' \, b^{2i+1}(2i+1+1)...(2i+1+a)u^{2i+1+a}] \bmod 68 = R_i' - 68K_i' \,,$$

where $R_i' = M_i' \, b^{2i+1}(2i+1+1)...(2i+1+a)\, u^{2i+1+a}$ and $K_i' = \dfrac{R_i' - M_i''}{68}$.

If we apply Natural transform successively k times, results can be generalized as

$$M_i^k = [M_i^{k-1} \, b^{2i+1}(2i+1+1)...(2i+1+a)u^{2i+1+a}] \bmod 68 = R_i^{k-1} - 68K_i^{k-1} \,,$$

where $R_i^{k-1} = M_i^{k-1} \, b^{2i+1}(2i+1+1)...(2i+1+a)\, u^{2i+1+a}$ and $K_i^{k-1} = \dfrac{R_i^{k-1} - M_i^k}{68}$.

2.2 Decryption

This ciphertext message is again converted to numerical form by the intended receiver using the table.

$M_0^{'}$	$M_1^{'}$	$M_2^{'}$	$M_3^{'}$	$M_4^{'}$	$M_5^{'}$	$M_6^{'}$	$M_7^{'}$	$M_8^{'}$
20	48	52	40	12	60	52	36	24

Let $R_i = M_i' + 68K_i$,

$$\begin{aligned} M\ u^2(-\frac{d}{ds})^2 \frac{3u}{s^2-3^2u^2} &= 1584\frac{1}{s^4} + 691200\frac{1}{s^6} + 52254720\frac{1}{s^8} + 2902376448\frac{1}{s^{10}} \\ &+ 2.30578\times10^{11}\frac{1}{s^{12}} + 2.26385\times10^{11}\frac{1}{s^{14}} + 2.85246\times10^{14}\frac{1}{s^{16}} \\ &+ 1.43237\times10^{16}\frac{1}{s^{18}} + 8.33604\times10^{17}\frac{1}{s^{20}}. \end{aligned}$$

After applying inverse Natural transform and replacing back $u=2$, we get

$$\begin{aligned} Mt^2\sinh(3t) &= 11\frac{3t^3}{1!} + 40\frac{3^3t^5}{3!} + 40\frac{3^5t^7}{5!} + 36\frac{3^7t^9}{7!} + 52\frac{3^9t^{11}}{9!} + 1\frac{3^{11}t^{13}}{11!} \\ &+ 26\frac{3^{13}t^{15}}{13!} + 28\frac{3^{15}t^{17}}{15!} + 36\frac{3^{17}t^{19}}{17!}. \end{aligned}$$

We, successfully, recover the message as

11	*40*	*40*	*36*	*52*	*1*	*26*	*28*	*36*
L	***o***	***o***	***k***		***B***	***a***	***c***	***k***

Hence, generalized iterative formula for decryption process can be formulated as

$$M\ u^n(-\frac{d}{ds})^n \frac{bu}{s^2-b^2u^2} = \sum_{i=0}^{m} \frac{R_i^{k-1}}{s^{(2i+1+a)+1}} \,,$$

where $M_i^{k-1} = \dfrac{68K_i^{k-1} + M_i^k}{b^{2i+1}(2i+2)...(2i+a)u^{2i+1+a}}$ and $R_i^k = 68K_i^k + M_i^k$.

CONCLUSION

The motivation of this work is to provide more secure techniques that may prove helpful in limiting the activities of hackers by the use of complex and generalized functions while encoding and decoding the data. The use of Natural transform instead of Laplace transformation, definitely, provides stronger encryption due to an additional parameter *u*. Moreover, by using this iterative technique, multiple ciphertexts can be formulated by using same plaintext just by changing values of parameters $(a,b), u$ and number of iterations. Generalized form of cipher texts and plaintexts can also be established by using power series of cosine hyperbolic function instead of sine hyperbolic function.

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APPLICATION OF CALCULUS AND LINEAR ALGEBRA IN DEEP LEARNING FOR FRUIT IDENTIFICATION FROM ITS IMAGE

Amina Firdous, Ghulam Farid, Muhammad Adil Zeeshan
Kiran Shahzadi and **Zainab Javid**
Department of Mathematics and Statistics
University of Agriculture, Faisalabad, Pakistan
Email: aminafirdous414@gmail.com
ghulamfarid@uaf.edu.pk
aadill893@gmail.com
kiranshahzadi4smd@gmail.com
zainikhilji98@gmail.com

ABSTRACT

Deep Learning is a part of machine learning and is based on artificial neural networks in which multiple layers of processing are used and progressively higher-level features are extracted from given data. Deep learning methods for image identification can now extract the number, pixel location, scale, and other features of an image from a database automatically. In this paper, a technique for fruit identification, from its image, is presented using deep learning. The technique is based on tools of Calculus and Linear Algebra. Python programming language in Anaconda software is utilized to develop a convolutional neural network for the identification. Anaconda supported libraries as Tensor flow and Keras are used for deep learning model implementation on Python Spyder IDE. Activation functions and cost functions are implemented by importing the above stated libraries.

Mathematics Subject Classification: 65K10, 05C50, 92B20

KEYWORDS

Deep learning, Convolutional neural network, Image identification, Activation functions.

1. INTRODUCTION

Artificial intelligence (AI) is the impersonation of human intelligence in machines that have been instructed to think and behave as humans. Deep learning is an artificial intelligence (AI) function that imitates the data processing and pattern creation processes of human brain. Recently, the popularization of deep learning algorithms based on linear algebra, calculus and probability theory have injected more possibilities into image processing filed. Image identification by deep learning is an interesting topic due to several applications in Agriculture, Medical, Self-Driving Cars, **Virtual Assistants, Visual Recognition.** In this work, neural networks will be used for the said purpose with the collaboration of linear algebra, calculus, and statistics.

Bongulwar *et al.* (2021) have proposed the idea of deep learning to create a model for the identification and classification of fruits and an automatic feature extraction method utilizing convolutional neural networks have also been created. Chung *et al.* (2019) have briefly discussed the uses of deep learning (DL) for fruit recognition and many other applications. Also, using Fruit 360 dataset, a simple overview of the Efficient Net architecture and convolution neural networks (CNNs) for fruit recognition have been provided. Sun *et al.* (2021) have presented a novel continual multiview task learning model called deep continual multiview task learning (DCMvTL), which combines deep matrix factorization and sparse subspace learning into a single framework.

Muresan *et al.* (2018) have proposed a new series of photos containing well-known fruits. Fruits-360 was the name of the dataset. The outcomes of a numerical experiment for training a neural network to detect fruits have also been described. Sa *et al.* (2016) have used deep convolutional neural networks to demonstrate a revolutionary technique to fruit detection and wanted to develop a fruit identification system that is accurate, rapid, and reliable, as well as a fundamental component of an autonomous agricultural robotic platform for fruit yield estimation and automated harvesting. This technique has been extended for the problem of fruit detection employing vision from two modalities: color (RGB) and near-infrared (NIR) through transfer learning (NIR). We refer Kelleher (2019), Ognjanovski (2019), Raschka (2015) and Skansi (2018) for the study of deep learning and neural network for fruit identification.

1.1 Image

An image (from Latin: imago) is a visual product, like a photograph or two-dimensional picture, that matches a subject, usually a physical item, and hence serves as a representation of it.

1.2 Deep Learning

Deep learning is a type of artificial intelligence (AI) which simulates how the human brain processes information and creates patterns in order to make assumptions. Deep learning is a subtype of AI that uses neural networks to understand uncontrolled from unstructured or unlabeled information.

1.3 Deep Learning Algorithms

Machine Learning is a sub-field of Artificial Intelligence, and Deep Learning is a sub-field of Machine Learning. It's a set of approaches for simulating data abstractions at a high level. Deep learning is a notion that was first introduced in the 1980s. It has only recently gained popularity due to two factors: it requires many labelled data and a lot of processing power. Natural language processing, photo categorization, and information retrieval are just a few of the deep learning applications that have grown in popularity in the last decade. Machine Learning is a branch that includes a variety of techniques and strategies for automatic identifying from available data and applying that learning to future predictions and forecasts.

The most widely used algorithm in Machine Learning and deep learning is the Artificial Neural Network, which is inspired by the humanoid brain. It is made up of neurons, which are integrated processing units. The input, hidden, and output layers make up an ANN. The input layer receives an input, such as a photo, and sends it towards the hidden layer, which subsequently returns output - the most likely item in the image. For more sophisticated functions, we can have numerous hidden levels.

1.4 Models of Deep Learning (artificial) Neural Network

1.4.1 Convolutional Neural Networks (CNNs)

Convolutional neural networks are titled after the mathematical operation convolution: The convolution of a function f with another function g is defined as

$$(u * v)(t) = \int_{-\infty}^{\infty} u(x)\, v(t-x)\, dx, \tag{1}$$

and

$$(u * v)(t) = \sum_{X=-\infty}^{\infty} u(x)\, v(t-x) \tag{2}$$

in the continuous or discrete domain respectively. Here *u* is the pixel's intensity and *v* are a two-dimensional weighting function known as kernel, convolution is frequently used in image processing.

Convolutional neural networks (CNN) are a type of deep learning model. In this network, loss layers, pooling layers, convolutional layers, fully connected layers, and ReLU layers, are all present. In a conventional neural networks CNN, every convolutional layer is passed by a Rectified. Linear Unit (ReLU) layer, a Pooling layer, one or even more convolutional layers, and then fully connected layers. Each layer of a CNN network will be described in this section.

1.4.2 Convolutional Layers

Convolutional layers are titled due to convolution operation. Convolution is a mathematical process that involves two main functions and provide a new third function that is a modified form of one of them. The new function will return the integral of the two point-wise multiplication functions as a function of the number by which one of the original functions is translated. A convolutional layer is made up of kernels, which are groups of neurons.

The produced activation maps are combined to produce a convolutional layer's output, which would then be utilized to determine the next layer’s input. A convolutional layer is applied to a 32 X 32(say) image to create a 28 X 28 (say) Feature map. If we add more convolutional layers, the dimensions will be further lowered.

1.4.3 Pooling Layers

Pooling layers being used to reduce the spatial dimensions of the image as well as the number of nodes undertaken in the network. 2x2 filters with a stride of 2" are employed in some of the most popular pooling layer. As a result, the input is reduced to a small percentage of its original size.

1.4.4 Fully Connected Layers

Fully connected layers are those that are fully connected in a conventional neural network. Every output of the preceding layers is associated with each neuron in the fully linked layer.

1.4.5 Loss Layers

When a network differs from its actual outcome, loss layers are utilized to compensate it. This is usually the last layer of a network.

1.5 Activation Functions

There are different types of activation functions which can be used for neuron output activation. These are step function, sigmoid function, Softmax and rectified linear unit function etc. Cost function will be used to minimized the prediction error which will work on the principle of gradient descent method.

2. METHODOLOGY

2.1 Designed Algorithm

The steps of the algorithm are explained in the following self-explained flow chart.

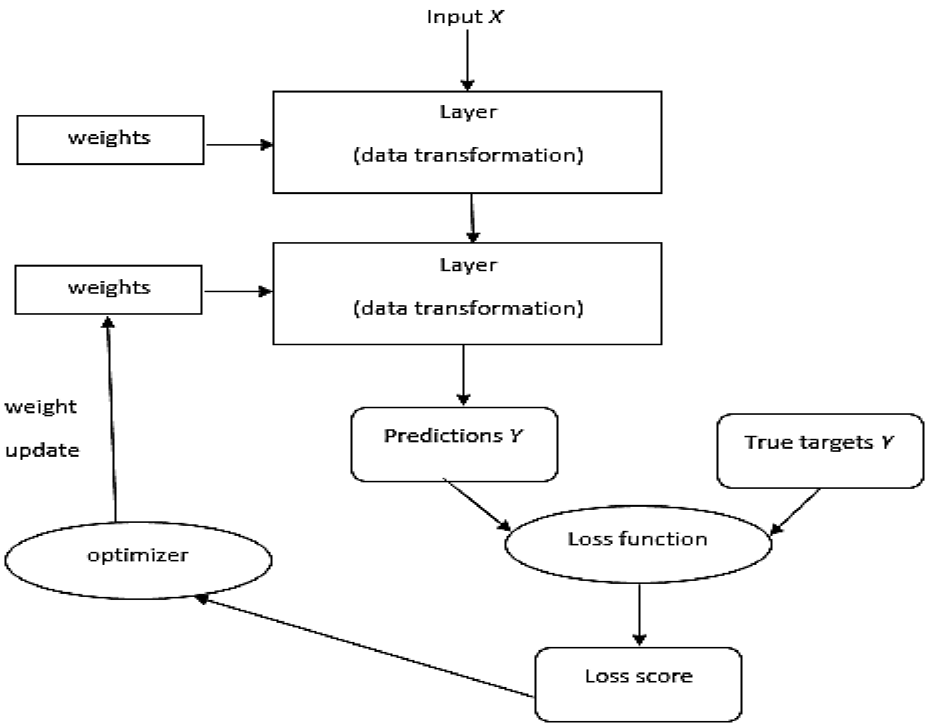

Figure 1: Proposed Algorithm for Fruit Identification

2.2 Steps for Building CNN model

Step 1 Convolution
Step 2 Max Pooling
Step 3 Flattening
Step 4 Full Connection

2.3 Dataset

The fruits 360 dataset provided all of the photos used in training and testing. There are 1212 different fruits photographs in the dataset, divided into 6 classes. To create the model, we used 75% of the images from the training set and 25% of the images from the testing set.

Table 1
Using Categories of Fruits

Name of Fruits	Number of Training Images	Number of Testing Images
Fresh Apple	186	46
Fresh Banana	175	43
Fresh orange	165	41
Rotten Apple	263	65
Rotten Banana	245	61
Rotten Orange	178	44

3. EXPERIMENTAL RESULTS

In this paper, On the Fruit Dataset, we used Keras and the Tensorflow library to identify the network's higher classification performance. We took 1212 photos from the Fruits 360 dataset, dividing them into 6 groups: 75% of the photos are treated for training, while 25% are treated for testing the model. With a batch size of 12, the network is trained for 20 epochs. When we matched our model to current state-of-the-art models, the outcomes were outstanding. The proposed model's accuracy was 95.67 percent. The outputs of our model are very excellent and suitable for usage in real-world applications when compared to the results of conventional models. The machine's overall efficiency in fruit identification will be improved as a result of the increased accuracy and precision.

Table 2
Using Categories of Fruits

Name of Fruits	Number of Training Images	Number of Testing Images	Accuracy on Training Set	Accuracy on Testing Set
Fresh Apple	186	46	90%	84%
Fresh Banana	175	43	90.1%	87%
Fresh Orange	165	41	90.6%	89%
Rotten Apple	263	65	94%	84.5%
Rotten Banana	245	61	89%	86%
Rotten Orange	178	44	97%	81%

4. COMMENTS AND CONCLUSION

This paper explores the identification of fruits using Python with the Keras and TensorFlow libraries. Throughout the experiment, the rate of identification has improved significantly. At epoch 20, the model had the best accuracy rate of all the examples, with a score of 98 percent. The machine's overall performance in terms of fruit identification will improve as a result of this increased accuracy.

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APPLICATION OF DEEP DREAM LEARNING IN ISLAMIC ART AND DESIGNING

Abdul Ghafoor, Muhammad Nabeel, Fatima Ibrahim, Emaan Baig, Gohar Ali Hayat, Umaira Saeed, Waqas Ahmed and **Danish Hassan**[§]
Department of Textile & Clothing, National Textile University, Karachi, Pakistan.
Email: abdulghafoor11194@gmail.com
nabeelarif12369@gmail.com
sheikhhfatimaibrahim@gmail.com
emaanbaig465@gmail.com
goharali1994@yahoo.com,
umairasaeed@ntu.edu.pk
waqasahmed1910@outlook.com
danishhassan@ntu.edu.pk
[§]Corresponding author

ABSTRACT

State-of-the-art fashions in image-based 3D object modernization in the deep learning era 3D rebuilding is a long-standing ill-posed topic that the computer vision, computer graphics, and machine learning fields have studied for decades. Image-based 3D modernization using convolutional neural networks (CNN) has piqued curiosity and exhibited remarkable performance since 2015. The research work represents a detailed overview of current developments in the discipline, keeping in view the new era for rapid growth. Particularly interested in works that employ deep learning algorithms to predict the 3D geometry of generic items from a single or several RGB photos. The objective of this research work is to apply deep dream learning on different Islamic art to produce different unique designs.

KEYWORDS

Artificial Intelligence, Machine learning, Deep dream learning, Art and design.

1. INTRODUCTION

The Islamic art is about 1400 years back came into existence. İt is observed arround the world where Muslims lives It is not as an art for specific region. Islamic art do not contains figurative but it includes the shapes of different geometry composed of a design in repetitive. Islamic art consists of intricate designs of geometric floral pattern and motifs, vegetal patterns, calligraphy and many other elements [1].

Islamic art can be observe on the Minarets or towers, domes, palaces, mosques, mihrab, graveyards, museums, on many ceramic objects such as tiles, utensils, and pottery items, and even on private homes [2].

Geometric pattern have simple rectangle, square, pentagon, octagon, hexagon, and other inorganic shapes. As a result Geometry plays a big role in Islamic art.

Vegetal patterns are all designs related to plants. Geometric patterns and vegetal pattern are incorporated to form a single beautiful Islamic motif or design [3 & 6].

Deep Dream Learning is the application of Artificial Intelligence (AI). There are different types of AI but in this research work we have incorporated the two types: Deep Dream AI and Deep Dream Scope. Deep Dream AI is one of the technique that can be use in art and designing to painting and different images. Deep Dream Scope is another application of AI that can combine the two or more images to generate a new one very cleanly and beautyfully [4 & 5].

2. METHODOLOGY

We have visited the Makli Graveyard and collected the pictures. The Deep Dream Learning technique of Artificial Intelligence was used on the pictures to generate new images.

Deep dream is one of the more intriguing deep learning applications in computer vision. It is, in fact, a computer program created by Google's Alexander Mordvintsev.

Deep Dream AI basically is a very useful application that can be used to generate art and new picture. Basically this application process the two or more pictures to create a new picture having the flavors of all the used.

Deep dream Scope basically two different faces are combine to create a new face is provide is called Deep Dream Scope [1 & 6].

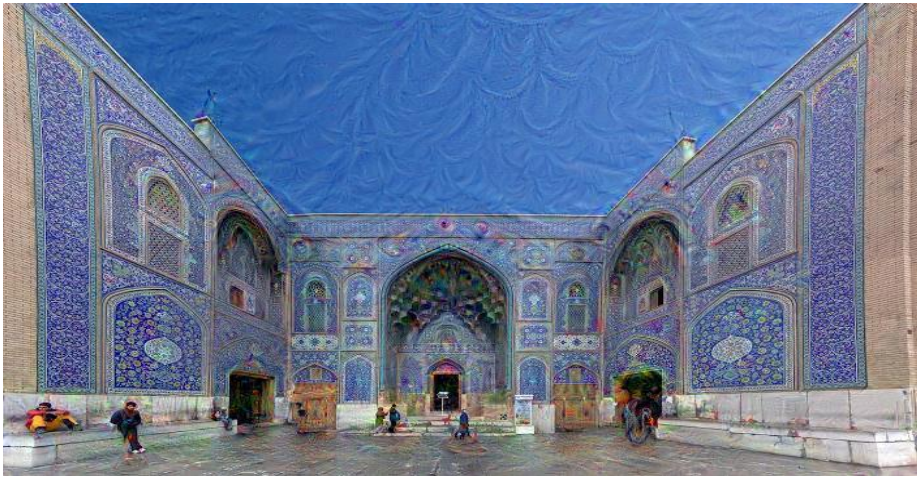
3. RESULTS AND DISCUSSION

Islamic Art and Designing

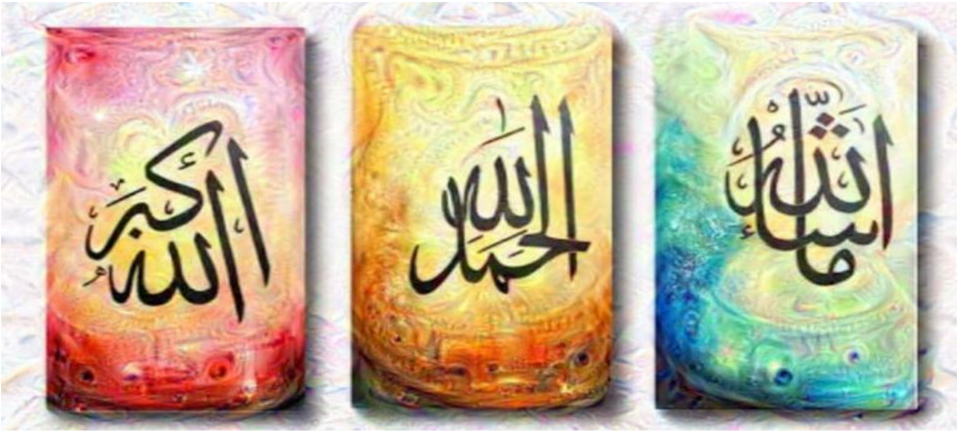
In Islamic art, there are repeated motifs, such as the arabesque, which is a repetition of stylized, geometrical floral or vegetal designs. In Islamic art, the arabesque is frequently employed to represent God's transcendent,indivisible, and endless essence.

Application of Deep Dream Learning in Islamic Art and Designing

Islamic art is often dynamic and specific. Unlike Christian art, Islamic art is not limited to religious work, but includes all artistic traditions of Muslim culture. Like we ever think of a design or art but it is not so easy to make. So with Deep Dream Learning we can create our own design and art. Without any hindrance. If we have to do Islamic art on a tree, it is very difficult to draw a picture with its hands. And if we combine a picture of a tree and an Islamic art in Deep Dream, it will easily become a beautiful picture and then we can make it.



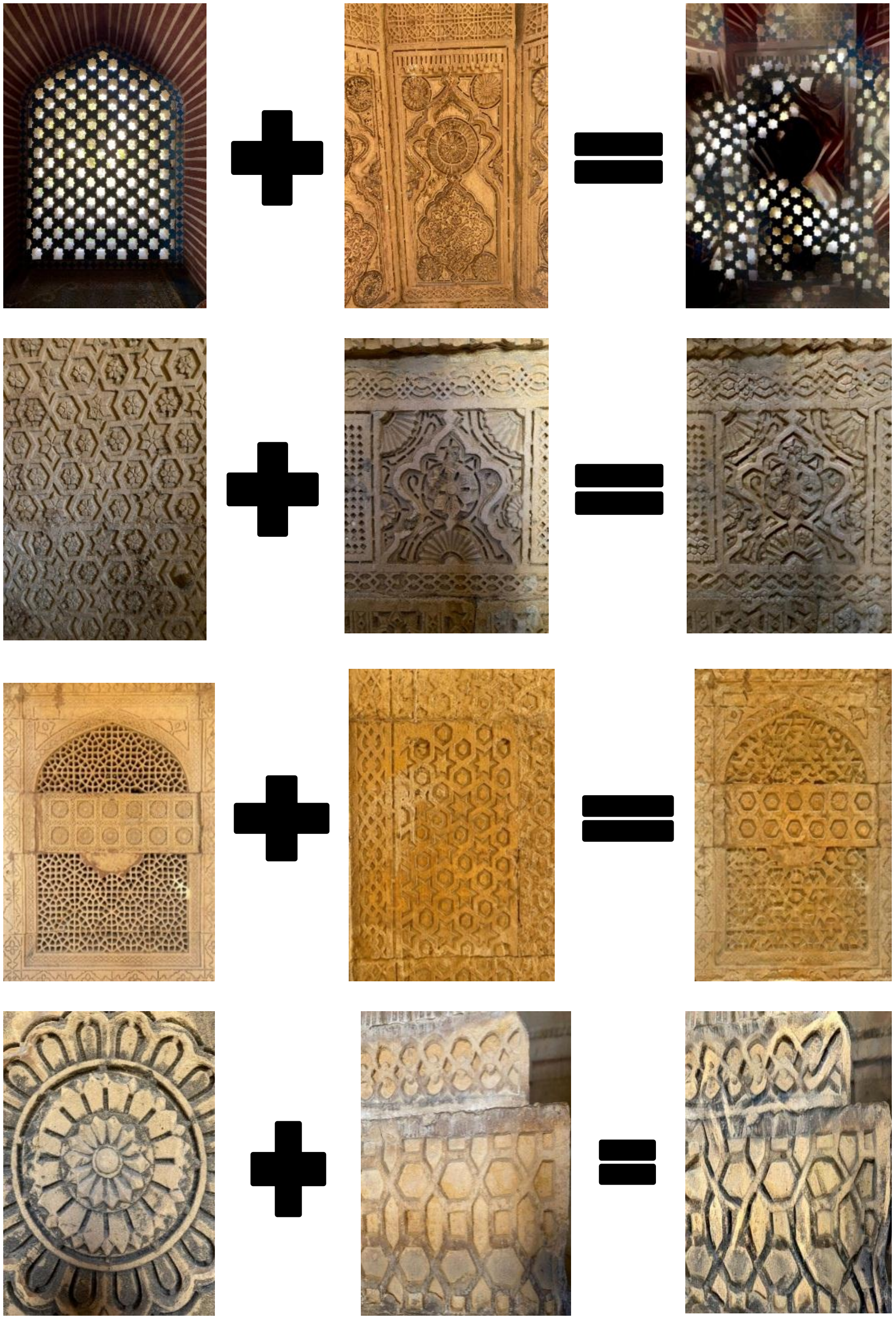
DIFFERENCE



Makli Graveyard

Makli graveyard history in Urdu. Makli ka graveyard (Makli Necropolis) is one of the largest (graveyard) funerary sites in the world, spread over an area of 10 kilometers near the city of Thatta, in Sindh Pakistan. A capital of four successive dynasties. The site houses approximately 500,000 to 1 million tombs built over 400 years period. Makli graveyard is declared a world heritage site.

Old Islamic art includes in the graveyard. Since the Makli graveyard has its worldwide importance in the perspective of ancient Islamic art and calligraphy. The collected pictures were mixed to create the new pictures. For this purpose, we have used Deep Dream AI and Deep Dream Scope.



4. CONCLUSION

The technique Deep Dream Learning is very beneficial to designing new pictures. The used tool was applied to the Islamic beautiful Art and new pictures were obtained by combining the selected pictures. These generated pictures can be used in new constructions and designing on different buildings.

5. ACKNOWLEDGEMENT

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STATISTICAL AND REGRESSION ANALYSIS OF PAKISTAN TEXTILE RELATED EXPORTS IN DIFFERENT CONTINENTS (2003-2019)

Muhammad Ebad, Kashaf-Ul-Hassan, Waqas Ahmed, Sohail Ahmed
Hamza Ashraf, Moiz Ali and **Danish Hassan**§
Department of Textile & Clothing, National Textile University, Karachi, Pakistan.
Email: muhammadebad18@gmail.com
kashafulhassan345@gmail.com
waqasahmed1910@outlook.com
sohailahmed.ntu@gmail.com
tip1.hamzaashraf@gmail.com
moizali0298@gmail.com
danishhassan@ntu.edu.pk
§Corresponding author

ABSTRACT

In this research paper the exploration and research work contains the statistical as well as regression analysis with the future predictions in textile industry exports (2022-2030) for different regions of the global. Up till now, the highest textile exports of Pakistan were done with the regions like East Asia and pacific, Europe and central Asia, Latin America and Caribbean, Middle East and North Africa, North America, south Asia and Sub-Saharan Africa. The objective of the research work to apply some statistical technique for descriptive analysis and forecasting on the selected time-series data sets (2003-2019) of Textile & Clothing related data. Minitab software was used to obtain the results and graphs.

KEYWORDS

Textile Export, Statistical Analysis, Regression Analysis.

1. INTRODUCTION

The objective of any country is to enlarge the sector which contributes a significant share in export, GDP and providing different opportunities of employment [3].

For the development of economically, Textile & Clothing is the major part of every industrialized and as well less developed countries, it plays a vital role to increase the export of country in all over the world. According to estimate that the larger export of textile in America and Europe. By removal of MFA in 2005, Pakistan will take an extra 62 to 67% other market access for textile and clothing [4].

Pakistan is one of the greatest producer and exporter of cotton yarn and its manufactured products. Definitely, it is the 5th largest manufacturer of cotton products as well as it is also 3rd largest cotton yarn producer in the world [1].

Pakistan's Textile Industry supplies 9% of the globally Textile businesses as per need or requirements of different customers, located primarily and mainly in Arab Countries, some countries of continent Africa, East Europe, Canadian markets, Latin America and also some countries of Southern Asia [2 & 7].

The aim of this study is to understand the trade prospects in the international market for Pakistan's textile and clothing exports [5].

2. METHODOLOGY

Simple linear Regression (SLR)

It is a statistical technique for Observation, summarizing and analyzing of relationships between two continuous or quantitative variables:

The explanatory, independent or predictor variable is designated or denoted by the X. The outcome, dependent or response variable is another variable denoted by Y. The equation of SLR is as under:

$$Y = a + bX \tag{1}$$

To summarize and observe manually for less number of data, the use of the appropriate model and technique of a and b may be beneficial [6]. All the results are obtained using the Minitab software.

3. RESULTS AND DISCUSSION

DESCRIPTIVE ANALYSIS OF DATA

We have applied some statistical methods to analyze the variations, Standard deviation, Linear Regression, Coefficient of Variation and Central tendency.

Table 1
Descriptive Statistics by using Statistical Tool

Formulas	PARTNERS (CUSTOMER) NAME						
	East Asia & Pacific	Europe & Central Asia	Latin America & Caribbean	Middle East & North Africa	North America	South Asia	Sub-Saharan Africa
Mean	1,767,385	4,695,472	333,171	688,145	3,324,063	613,203	302,835
StDeviations	587,544	1,285,301	95,520	143,742	370,745	215,184	33,841
Variance	345,208,218,025	1,651,997,672,372	9,124,037,597	20,661,790,919	137,451,920,409	46,304,112,619	1,145,189,111
Cvariations	33	27	29	21	11	35	11
Quartile 1	1,293,892	3,565,753	286,359	585,053	3,057,790	432,698	278,246
Medians	1,571,710	4,323,540	343,762	679,571	3,341,694	709,275	306,107
Quartile 3	2,198,646	5,739,882	419,562	734,457	3,530,945	796,620	330,185
IQR	904,754	2,174,129	133,203	149,404	473,155	363,923	51,940
Minimum	1200081.876	2651702.33	142665.427	538540.9345	2642971.925	199373.133	247362.679
Maximum	3008865.656	6725186.152	450915.812	1135888.632	4137388.435	840779.419	359366.064
Range	1808783.78	4073483.822	308250.385	597347.6975	1494416.51	641406.286	112003.385
Skewness	1.128662285	0.130267342	-0.648662564	2.017853825	0.261974843	-0.694123658	-0.030663248
Kurtosis	0.174334192	-1.161358387	-0.346284067	5.418281997	0.436962547	-0.902270252	-0.831241004
N for Mode	17	17	17	17	17	17	17

The Table 1 depicts the statistical analysis of all the continent. Less variation was observed for Sub-Saharan Africa. The maximum value depicts the more export for the Europe & Central Asia. The value of IQR endorses it and furnished in Table 1.

Table 2
Regression Analysis of the given Data

Year	Regression Equations						
	- 90798219 + 46030(Year)	- 4.88E+08 + 244910 (Year)	- 16035272 + 8139 (Year)	38750447 - 18927 (Year)	6755862 - 1707 (Year)	- 73545193 + 36876 (Year)	- 5740400 + 3005 (Year)
2020	2,182,381	6,718,200	405,508	517,907	3,307,722	944,327	329,700
2021	2,228,411	6,963,110	413,647	498,980	3,306,015	981,203	332,705
2022	2,274,441	7,208,020	421,786	480,053	3,304,308	1,018,079	335,710
2023	2,320,471	7,452,930	429,925	461,126	3,302,601	1,054,955	338,715
2024	2,366,501	7,697,840	438,064	442,199	3,300,894	1,091,831	341,720
2025	2,412,531	7,942,750	446,203	423,272	3,299,187	1,128,707	344,725
2030	2,642,681	9,167,300	486,898	328,637	3,290,652	1,313,087	359,750

The Result of Simple linear Regression analyses for different region are demonstrated in Table 2. The regression model for each selected continent obtained and incorporated in the Table 2. All the values are expected and predicted from the fitted model. The expected values indicated the more export in future. The expected values has been calculated for the year of 2020 to 2030.

REGRESSION ANALYSIS

The continent wise regression analysis is as under along with the coefficient of determination (R^2). The value of R^2 representing the significance and adequacy of the regression model fitted on each continent. The model is more reliable as R^2 approaches to the 100.

East Asia & Pacific versus Years
The regression equation is
East Asia & Pacific = - 90798219 + 46030 (Year)
Value of R^2 = 15.7%

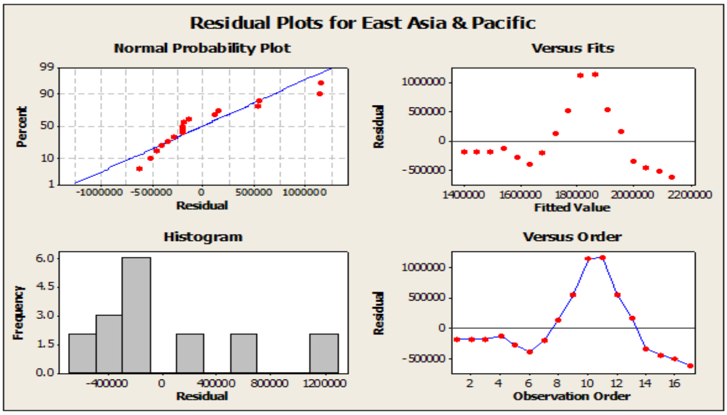


Figure 1: Residual Plots for East Asia & Pacific

Europe & Central Asia versus Years
The regression equation is
Europe & Central Asia = - 4.88E + 08 + 244910 (Year)
Value of R^2 = 92.6%

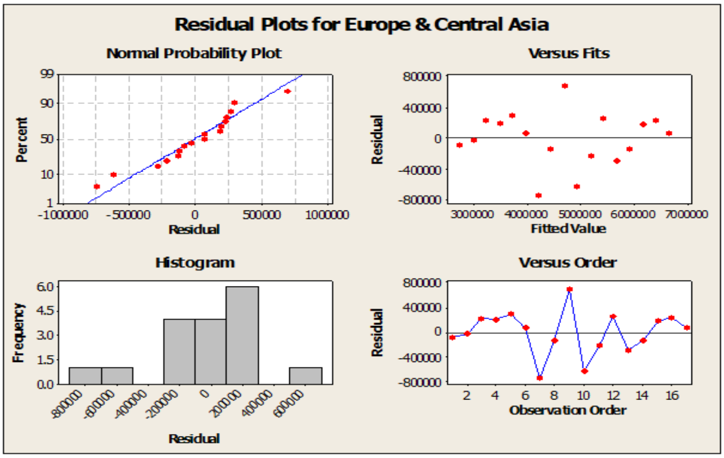


Figure 2: Residual Plots for Europe & Central Asia

Latin America & Caribbean versus Years
The regression equation is
Latin America & Caribbean = - 16035272 + 8139 (Year)
Value of R^2= 18.5%

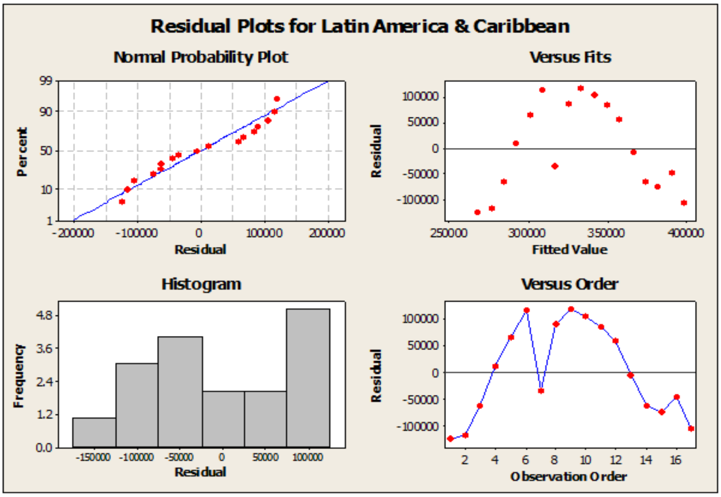


Figure 3: Residual Plots for Latin America & Caribbean

Middle East & North Africa versus Years
The regression equation is
Middle East & North Africa = 38750447 - 18927(Year)
Value of R^2= 44.2%

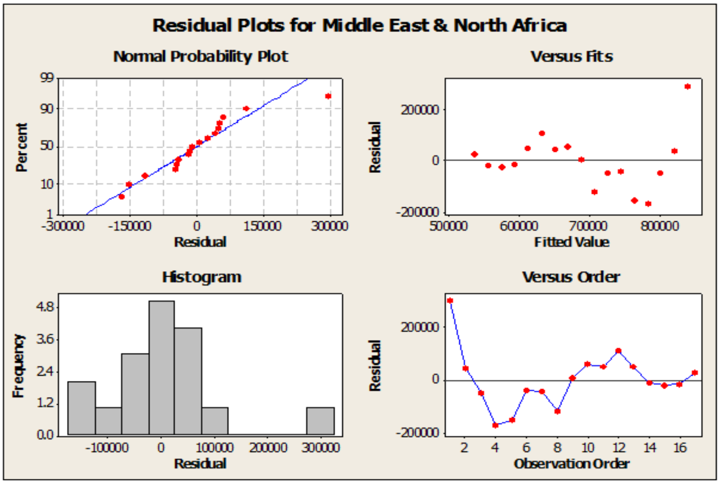


Figure 4: Residual Plots for Middle East & North Africa

North America versus Years
The regression equation is
North America = 6755862 - 1707(Year)
Value of R^2= 0.1%

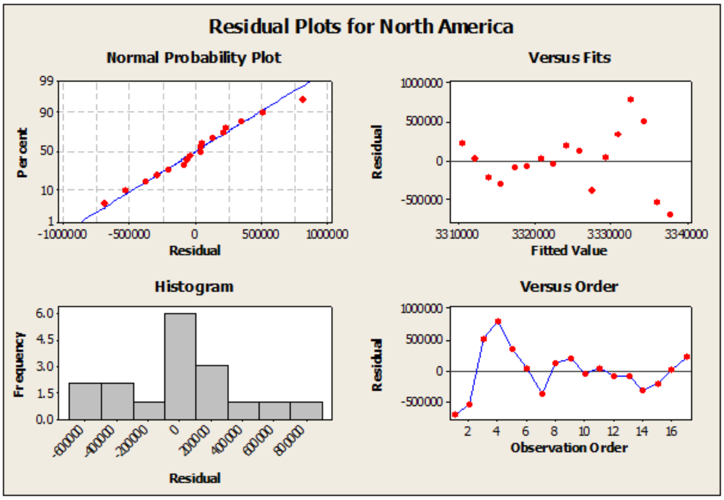


Figure 5: Residual Plots for North America

South Asia versus Years
The regression equation is
South Asia = - 73545193 + 36876 (Year)
Value of R^2= 74.9%

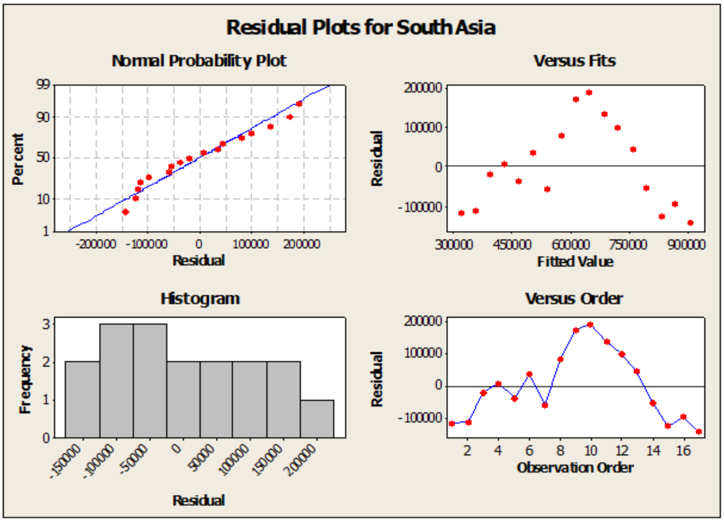


Figure 6: Residual Plots for South Asia

Sub-Saharan Africa versus Years
The regression equation is
Sub-Saharan Africa = - 5740400 + 3005(Year)
Value of R^2= 20.1%

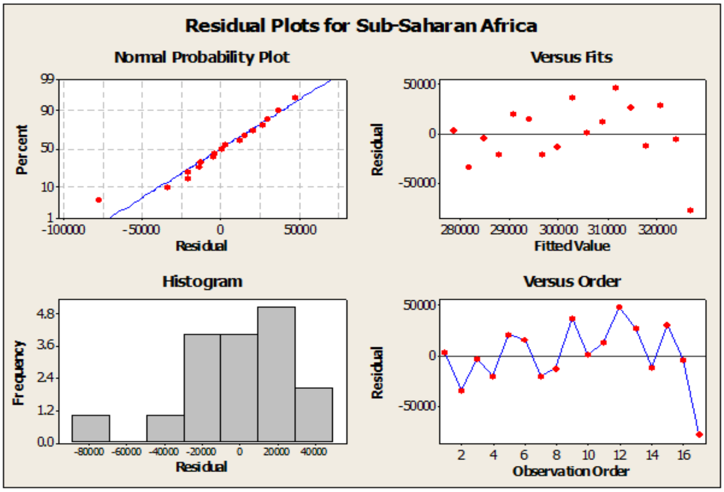
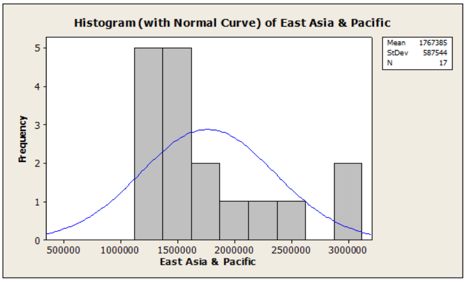


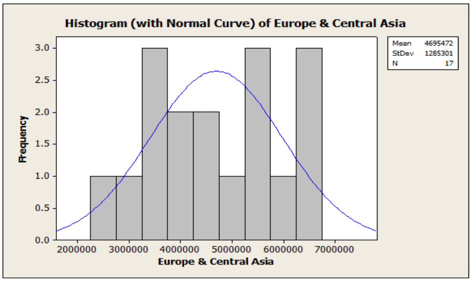
Figure 7: Residual Plots for Sub-Saharan Africa

Histogram (with Normal Curve):

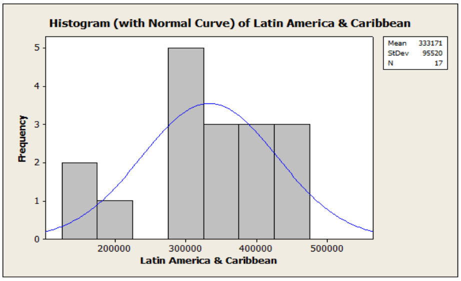
By this plot, normality and skewness of data can be observed. Also, these graphs show the increasing and decreasing phases of their respective average unit price.



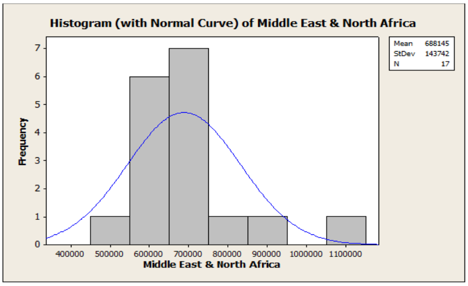
East Asia & Pacific



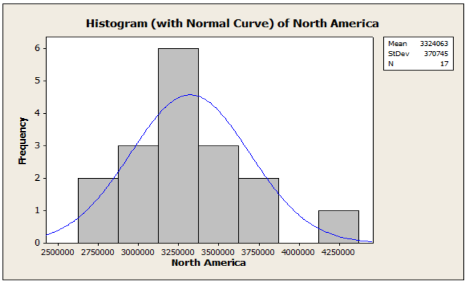
Europe & Central Asia



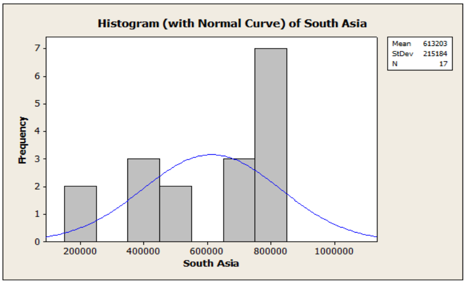
Latin America & Caribbean



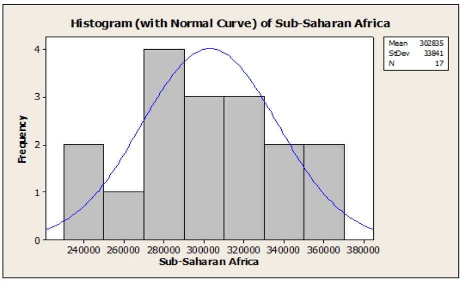
Middle East & North Africa



North America



South Asia



Sub-Saharan Africa

Figure 8: Histogram along with the Normal Fitted Curve

4. CONCLUSION

The Result is occurred that after applying the statistical tool, we can forecast the data for next days, moth, years etc. and can visualize the data in a graphically representation to understand easily. According to this given data (2003-2019) we have observed that the Export of textile and clothing is improving day by day. The historical time series data of some region shows that regression model is not best fitted since less value of R^2 but visual description analysis depicts that now Export is getting Stable in different Regions.

In regression analysis we used the export (US$ Thousand) of Textile and Clothing in all over the world of 2020 and expected value for the duration of 2019-2020 was found 14,405,745 whereas the observed export value was 12,526,537 for the same year (2019-2020). The model value was closed to the observed one.

5. ACKNOWLEDGEMENT

We are thankful to Dr. Danish Hassan for providing us with a great knowledge about statistical tool. This was a wonderful experience for us to learn and professionally develop ourselves during this mini-Project. We are thankful to attached website, where we got the data of textile & clothing Exports (2003-2019) in all over the world.

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STATISTICAL ANALYSIS OF PAKISTAN COTTON TEXTILE (1990 TO 2018)

Sohail Ahmed, Fatima Iftikhar, Muhammad Ibrahim, Aqil Muneer,
Bilal Randhawa, Ali Hassan, Sheraz Mughal and **Danish Hassan**
Department of Applied Sciences, National Textile University, Karachi, Pakistan
Email: sohailahmed ntu@gmail com
fatimaiftikhar0110@gmail com
muhammadibrahimshaikh03@gmail com
alih86105@gmail com
bilalrandhawa2017@gmail com
sheraz7860mughal@gmail com
aqilkhokhar83@gmail com
danishhassan@ntu edu pk

ABSTRACT

The performance of Pakistan textile industry particularly cotton textile sector that has been studied and analyzed over the past twenty eight years. The average value of Pakistan's cotton textile in unit products over twenty eight years from 1990-91 to 2017-18 has been discussed in this research work. In this regards, the data has been taken from Database Center of Textile Commissioner's Organization, Karachi. The statistical tools that include simple linear regression (SLR) and coefficient of variation (CV) have been used to analyze the time series data. Furthermore, through simple linear regression of garment and fabric AUV, we obtained a linear equation ($\boldsymbol{a}$ = -21.5 + 66.1 $\boldsymbol{b}$) where '$\boldsymbol{a}$' is the garment price and '$\boldsymbol{b}$' is fabric price. It was observed that how fabric average unit value affects the value price of garment. In addition, we have performed the descriptive analysis of selected data which has given us the highest CV of garment 'AUP that indicates garment price was not stable over past few years. The results depict the rise in annual inflation rate in textile sector for last few years. One of the reasons of inflation was the money invested in cotton farming. To reduce this inflation for some extent organic cotton should be produced as it is more sustainable. Organic cotton is grown devoid of pesticides and insecticides from the seeds that are not genetically modified. This will reduce harmful chemicals as well as it aims for environmental sustainability. It will also minimize the total farming cost by reducing the cost of chemicals and land damage due to chemicals.

KEYWORDS

Cotton, statistical tools, regression model, sustainability.

1. INTRODUCTION

Fabric.sector.be.present.the.foremost.industrial.area.that.is.playing.the.imperative.part in.the.financial.growth.of.Pakistan. Pakistan is one of greatest producer and exporter of cotton yarn and its products. Indeed, it is the 5th largest producer of cotton products as well as 3rd largest cotton yarn producer in the world [1] Average unit value (AUV) or average unit price (AUP) is the average price of a product in which it is sold, in a particular time period [2].

Raw cotton, cotton yarn, cotton fabric, towel and bed wear are calculated in terms of $/kg. Whereas, AUP of knitwear and garment have been shown in terms of $/doz.

By analyzing data of Pakistan cotton textile' average unit value, trend and variability in prices over certain time period can be calculated In textile statistics, it is necessary to use latest marketing strategies to increase production of cotton textile and to work on reducing the AUP of these products [2].

2. METHODOLOGY

The twenty eight years Average Unit Price data of time series of cotton textile incorporated in the research and we have applied following statistical tools on it.

- Central Tendency
- Coefficient of Variation
- Standard Deviation
- Simple Linear Regression

The central tendency is the statistical technique that provides precise description of the whole data. It contains three important measures, mean, median and mode. Furthermore, central tendency determines the normality of data around its mean value. By the results obtained, bed wear average unit value over time was found more normal than other. In this research, we have calculated mean, median, mode, quartiles, skewness along with other parameters that are presented in Table 1.

The ratio between standard deviation (σ) and mean (μ) is known as coefficient of variation (CV). Here σ is the degree of variation of used data set values from defined standard or mean. The higher coefficient of variation indicates greater dispersion of data. However, lower value of coefficient of variation shows more precision or reliability [5].

Mathematically,

$$CV = (\sigma / \mu) \times 100 \quad (1)$$

The analysis of linear regression was used for predicting the variable values that is based on a different variable value. Variable that has to be determined is called as dependent variable. While independent variable is the one that predicts the value of dependent variable [6].

Visual statistics is a way to instantly and easily get the overview of whole data. It is the comparison of data variables graphically. The histogram, pie charts and box-whisker plots of average unit price have been plotted.

- **Descriptive Statistics**

Table 1
The Descriptive Analysis of the Used Time Series Data

Variable	Mean	St.dev	Variance	Coefvar	Minimum	Q1	Median
Raw Cotton	1.4050	0.4570	0.2088	32.52	0.7000	1.0300	1.3750
Cotton Yarn	2.527	0.538	0.289	21.27	1.730	2.095	2.400
Cotton Fabric	0.9171	0.2124	0.0451	23.16	0.5800	0.7300	0.9400
Towels	4.1529	0.4298	0.1847	10.35	3.3900	3.7975	4.1600
Bed Wear	6.000	0.599	0.359	9.99	4.960	5.485	6.045
Knitwear	21.108	2.230	4.974	10.57	17.100	19.347	21.140
Garment	39.11	17.72	313.97	45.31	21.18	23.37	33.37

Above results show that the coefficient of variation of bed wear AUV is least i.e. CV bed wear = 9.99%, showing that it has highest reliability while CV of garment AUV is highest, i.e. CV garment = 45.31% indicating its highest variability.

Table 2
Analysis of the Quartiles, Skewness and Kurtosis

Variable	Q3	Max	IQR	Mode	Mode	Skewness	Kurtosis
Raw Cotton	1.6575	2.5300	0.6275	1.03	3	0.8	0.71
Cotton Yarn	2.857	4.110	0.763	2.09	2	0.98	1.37
Cotton Fabric	1.0725	1.2900	0.3425	0.98	2	0.14	-1.08
Towels	4.5075	4.8800	0.7100	3.68	2	-0.01	-1.12
Bed Wear	6.530	7.020	1.045	5.36, 6.05, 6.53, 6.75	2	-0.03	-1.25
Knitwear	23.075	25.070	3.727	-	0	-0.04	-1.04
Garment	61.09	68.78	37.72	-	0	0.71	-1.19

In the above continuous data of knitwear and garment there is no mode which means no value is repeated.

- **Regression Analysis: Garment (a) versus Cotton Fabric (b)**

The regression equation is

$$a = -21.5 + 66.1\,b \tag{2}$$

Table 3
Analysis Regarding Linear Regression Equation

Predictor	Coef	SE Coef	T	P
Constant	-21.510	9.388	-2.29	0.030
Cotton Fabric	66.096	9.981	6.62	0.000

S = 11.0162 R^2 = 62.8% R^2. adj. = 61.3%

Table 4
Analysis of Variance (ANOVA)

Source	DF	SS	MS	F	P
Regression	1	5321.8	5321.8	43.85	0.000
Residual Error	26	3155.3	121.4		
Total	27	8477.1			

With the help of above linear regression model, we can find how garment price will get affected when cost of cotton fabric rises or not.

- **Residual Plots for Garment**

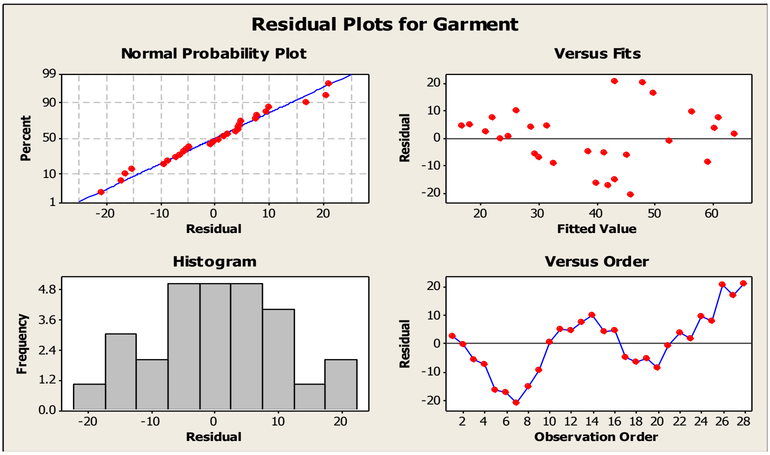

Figure 1: Normal Probability and Residual Plot

The visual statistical analyses are as under:

- **Histogram with Normal Curve:**

The graph indicated data set values around normal curve. By this, normality and skewness of data can be observed. Also these graphs show the increasing and decreasing trend of their respective average unit price.

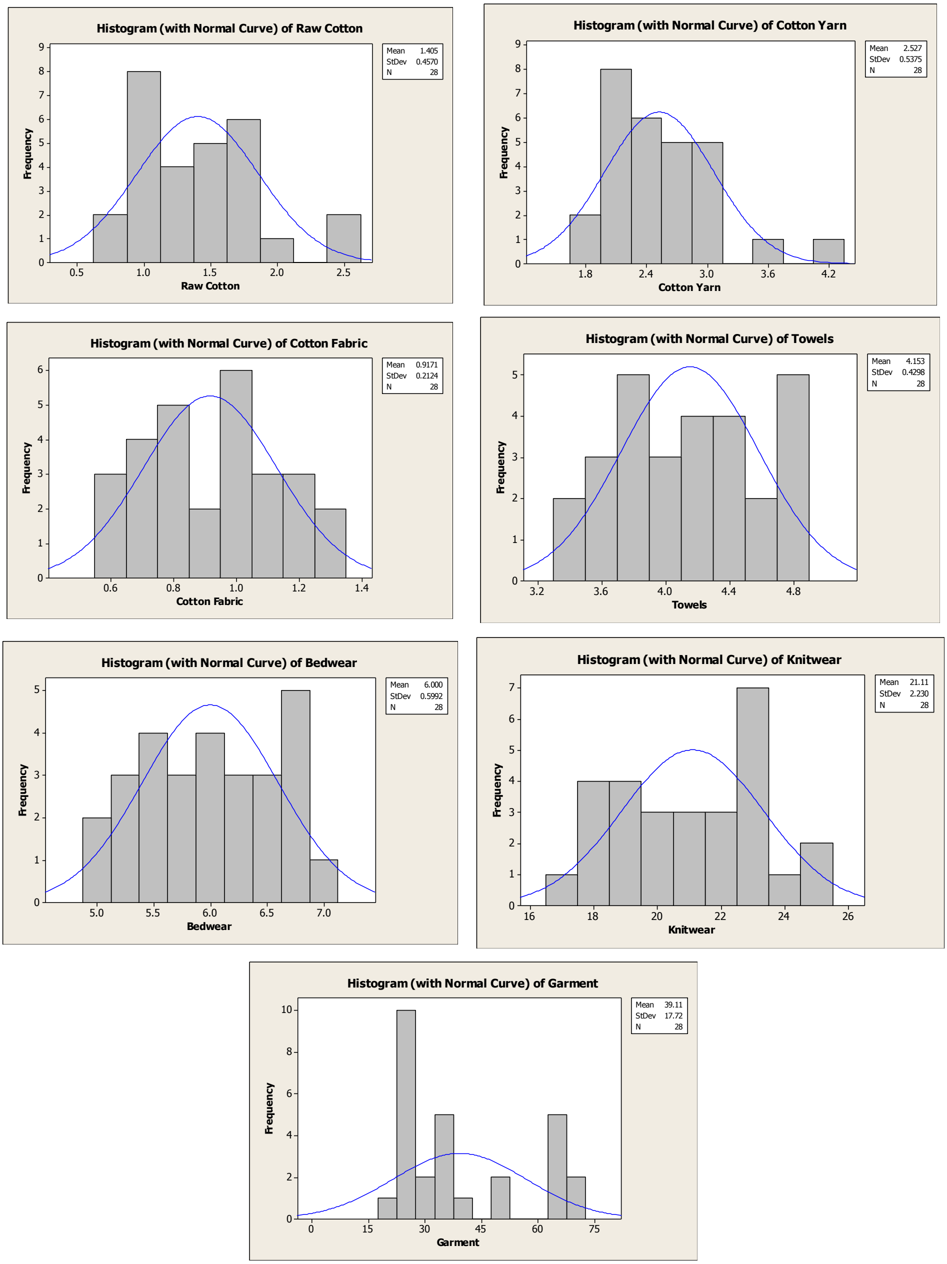

Figure 2: Histogram with Normal Curve Fitting

- **Box Plot**

Box plot displays the distribution of data and shows the mid-points and spread of whole data.

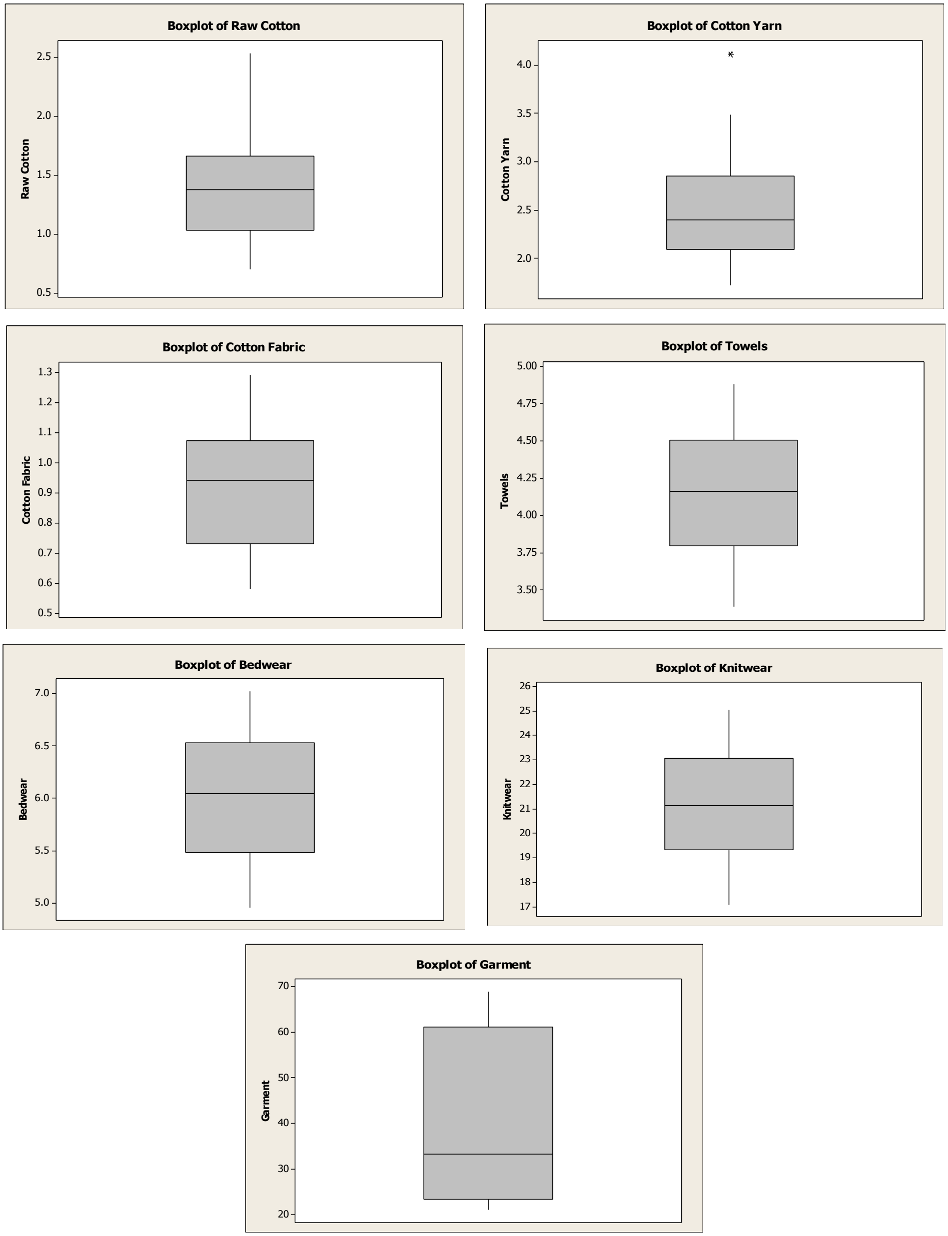

Figure 3: Box Plot Depicting the Variation in Selected Data Sets

More equal division of boxes shows that the corresponding data is more symmetrical. These plots are useful in the perspective of getting a quick summary of given data.

- **Time Series Plot**
  Time series plot has been used to show how changes occur in data over time.

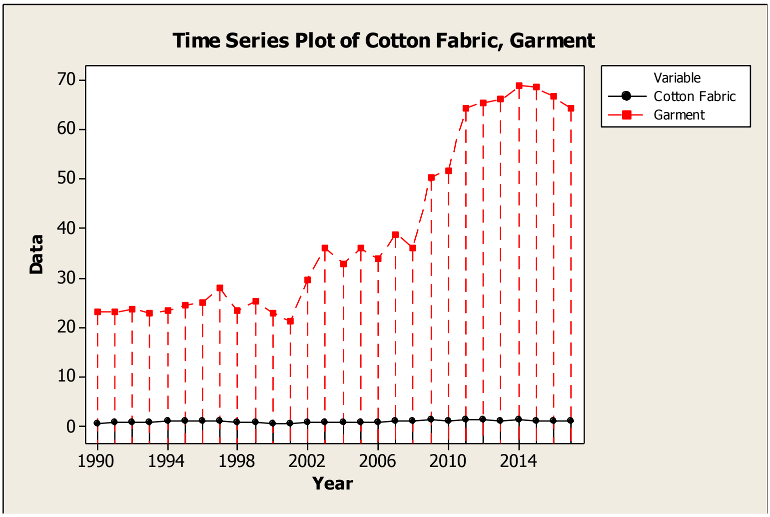

Figure 4: The Time Series Plot of Cotton Fabric and Garment

In the above graph it can be easily observed that cotton fabric AUV has been remain almost constant through the given years whereas garment AUV kept rising through these years.

- **Pie Chart**
  Pie charts have been used to represent the percentages of all data sets.

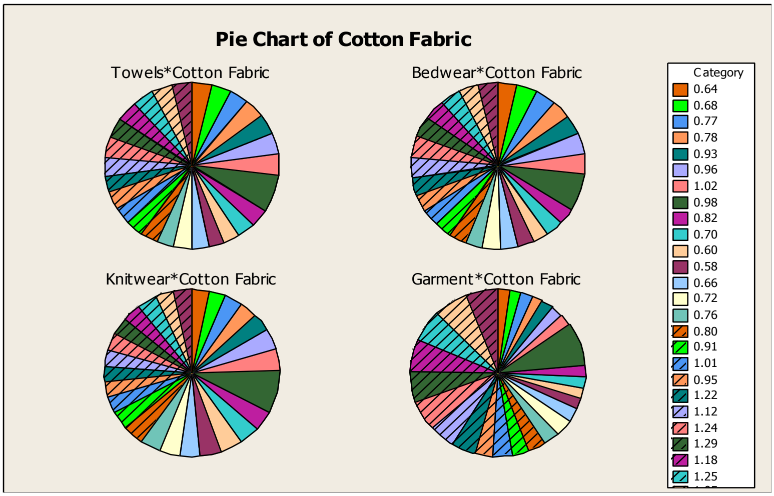

Figure 5: Pie Chart of Different Cotton Fabric

3. RESULTS AND DISCUSSION

By applying statistical tools on average unit value of Pakistan cotton textile products and comparing their values it has been observed that as the years passes the average unit price of these products increases. So according to the 28 years data analysis we can understand that AUV of garment has enhanced whereas cotton fabric itself shown lowest variability. Furthermore, the bed wear central tendency shows normal distribution with respect to the other related cotton textile products. Also the value of coefficient of variation of garment is highest as compared to remaining products. This huge increment in price of garment reflects more variation in Pakistan.

ACKNOWLEDGEMENT

We convey our heartfelt thanks to Allah ﷻ the most merciful and the most beneficent and the Holy Prophet Hazrat Muhammad ﷺ We express our sincere gratitude to Dr. Danish Hassan Asst. Prof. (Mathematics) National Textile University Karachi Campus as it would not be possible without his guidance and support. Lastly, we would like to thank our parents who encouraged us and prayed for us.

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IMPACT OF COVID-19 PANDEMIC ON MENTAL HEALTH OF YOUNG PEOPLE

Anam Zakir
Department of Statistics, Virtual University of Pakistan
Lahore, Pakistan
Email: anam.zakir@vu.edu.pk

ABSTRACT

Mental health conditions are increasing worldwide. Mainly because of demographic changes, there has been a 13% rise in mental health conditions and substance use disorders in the last decade. Around 20% of the world's children and adolescents have a mental health condition, with suicide the second leading cause of death among 15-29-year-olds. Approximately one in five people in post-conflict settings have a mental health condition. Despite these Figure, the global median of government health expenditure that goes to mental health is less than 2%. Youth mental health is a rapidly developing field with a focus on prevention, early identification, treatment innovation and service development. In this perspective piece, we discuss the effects of COVID-19 on young people's mental health. The psycho-social effects of COVID-19 disproportionately affect young people. Both immediate and longer-term factors through which young people are affected include social isolation, changes to the delivery of therapeutic services and almost complete loss of all structured occupations (school, work and training) within this population group. Longer-term mechanisms include the effects of the predicted recession on young people's mental health. In this study, primary data is used and being analyzed using statistical software, SPSS.

INTRODUCTION

Mental disorders are the largest cause of years lived with disability worldwide (White ford et al. 2015). Upto 80% of mental disorders first occur before the age of 26 (Caspi et al. 2020; Kessler et al. 2005). Earlier age of onset of mental disorder is associated with increased risks of development of co morbidity and persistence of mental health disorder to midlife (Caspi et al. 2020). Young people who remain free of mental disorder have longitudinally better outcomes (Caspi et al. 2020). Youth mental health problems cast a long shadow over adult health and psychosocial functioning. The magnitude of the effects of mental health problems in youth over the life course far surpasses the effects of early physical health problems (Goodman et al. 2011). In this paper, we will outline how youth, whilst less susceptible to severe COVID-19 infection, is more at risk of the negative psychosocial effects of the pandemic that was officially declare on 11 March 2020 (Holmes et al. 2020).

1.1 Path Physiology and Molecular Mechanism

Emerging and re-emerging infectious diseases are constant challenges for global public health. Recent cases of pneumonia in Wuhan, China, have led to the discovery of a new type of zoonotic corona virus an enveloped RNA virus, commonly found in humans, other mammals and birds, capable of causing respiratory, enteric, liver and neurological disorders. Although COVID-19 has a low lethality of around 3%, its transmissibility is high, with respiratory secretions being the main means of spreading SARS-CoV-2.

One of the main concerns is the impact of this pandemic on health outcomes, especially on mental health. Overall, in the event of pandemics or natural disasters, people's physical health and the Figurant against the pathogen are the primary focus of attention of stakeholders/managers and health professionals, so the implications for mental health tend to be overlooked or underestimated. However, measures taken to reduce the psychological implications of the pandemic cannot be minimize at this time, mainly because the psychological implications can be more lasting and prevalent than the infection of COVID-19 itself, with repercussions in different sectors of society, resulting in important gaps in facing the negative issues associated with COVID-19.

Studies have suggested that the fear of being infect by a potentially fatal virus, of rapid spread, whose origins, nature and course are still little known, ends up affecting the psychological well-being of many people. Symptoms of depression, anxiety and stress in the face of the pandemic have been identifying in the general population. In addition, suicide cases potentially linked to the psychological implications of COVID-19 have also been report in some countries, for example, South Korea and India. In addition to the psychological implications directly related to COVID-19, measures to contain the pandemic may also consist of risk factors for mental health. In a review of the quarantine, researchers identified that the negative effects of this measure include symptoms of posttraumatic stress, confusion and anger. Concerns about the scarcity of supplies and financial losses also cause damage to psychological well-being.

In this context, it also tends to increase social stigma and discriminatory behaviors against some specific groups that are more vulnerable. The rapid spread of the new corona virus throughout the world, the uncertainties about how to control the disease and its severity, in addition to the unpredictability about the duration of the pandemic and its consequences are characterize as risk factors for the mental health of the general population. This scenario also seems to be aggravated by the spread of myths, fake news and misinformation about infection and preventive measures, as well as by the difficulty of the general population in understanding the guidelines of health authorities. Among the few population-based studies carried out to date about the implications of the COVID-19 pandemic on mental health, we highlight that the study held with the general population in China, including 1210 participants in 194 cities, stands out during the initial stage of pandemic. This study revealed moderate to severe symptoms of anxiety, depression and stress in 28.8%, 16.5% and 8.1% of respondents, respectively. In addition, 75.2% of respondents reported fear of their family members becoming infect with the new corona virus. Moreover, being a woman, student and having physical symptoms linked to COVID-19 or previous health problems, were factors significantly associated with higher levels of anxiety, depression and stress. The world's scientific community has been mobilizing in record time to disseminate knowledge about COVID19.

On 13 February 2020, the vocabulary COVID-19 had already been added to the Medical Subject Heading (MesH) terms as a subject heading index in Medical Literature Analysis and Retrieval System Online (MEDLINE) defined as 'A viral disorder characterized by high fever; cough; dyspnea; renal dysfunction and other symptoms of a viral pneumonia. A corona virus SARS-CoV-2 in the genus beta corona virus is the suspected agent'. Since the first scientific publications on COVID-19 so far, the MesH Term 'COVID-19' has been cited in 17301 publications on Pub Med. However, studies on the implications for the mental health of young people and adults because of the new corona virus pandemic are still scarce, as it is a recent phenomenon, but which point to important negative repercussions. Hence, following the Preferred Reporting Items for Systematic Reviews and Meta Analyses Protocols (PRISMA-P) checklist as guideline, we propose a systematic and a reproducible strategy to query the literature about the impact of the COVID-19 pandemic on the mental health of young people and adults.

1.2 Increasing Depression and Anxiety

After more than a year of lockdowns and losses, studies show that the pandemic is taking a massive toll on people's mental health. Especially young people are disproportionally affected and are more prone to depression, anxiety and substance abuse. The researchers analyzed data in 2-week periods between August 19, 2020 and February 1, 2021. They used the HPS rapid-response online survey developed by the United States Census Bureau and other federal agencies. The researchers evaluated self-reported data from adults age 18 years and older. Questions were based on the Patient Health Questionnaire (PHQ-4) for depression and anxiety. The survey included 2 phases and 790,633 individuals.

"Continued near real-time monitoring of mental health trends by demographic characteristics is critical during the COVID-19 pandemic," the researchers concluded. "These trends might be used to evaluate the impact of strategies that address mental health status and care of adults during the pandemic and to guide interventions for groups that are disproportionately affected.

1.3 How it Spreads

The virus that causes COVID-19 is mainly transmit through droplets generated when an infected person coughs, sneezes, or exhales. These droplets are too heavy to hang in the air, and quickly fall on floors or surfaces. You can be infecting by breathing in the virus if you are within close proximity of someone who has COVID-19, or by touching a contaminated surface and then your eyes, nose or mouth.

1.4 Symptoms

Most common symptoms are:

(a) Fever
(b) Dry cough
(c) Tiredness
(d) Less common symptoms
(e) Aches and pains
(f) Sore throat
(g) Diarrhea
(h) Conjunctivitis
(i) Headache
(j) Loss of taste or smell
(k) A rash on skin, or discoloration of fingers or toes

1.5 Preventions

Protect yourself and others around you by knowing the facts and taking appropriate precautions. Follow advice provided by your local health authority. To prevent the spread of COVID-19:

(a) Clean your hands often. Use soap and water, or an alcohol-based hand rub.
(b) Maintain a safe distance from anyone who is coughing or sneezing.
(c) Wear a mask when physical distancing is not possible.
(d) Do not touch your eyes, nose or mouth.
(e) Cover your nose and mouth with your bent elbow or a tissue when you cough or sneeze.
(f) Stay home if you feel unwell.
(g) If you have a fever, cough and difficulty breathing, seek medical attention.
(h) Calling in advance allows your healthcare provider to be quickly direct you to the right health facility. This protects you, and prevents the spread of viruses and other infections.
(i) Masks can help prevent the spread of the virus from the person wearing the mask to others. Masks alone do not protect against COVID-19, and should be combining with physical distancing and hand hygiene. Follow the advice provided by your local health authority.

1.6 Treatment

1. After exposure to someone who has COVID-19, do the following:
   (a) Call your health care provider or COVID-19 hotline to find out where and when to get a test.
   (b) Cooperate with contact-tracing procedures to stop the spread of the virus.
   (c) If testing is not available, stay home and quarantined for 14 days.
   (d) While you are in quarantine, do not go to work, to school or to public places. Ask someone to bring you supplies.
   (e) Keep at least a 1-meter distance from others, even from your family members.
   (f) Wear a medical mask to protect others, including if/when you need to seek medical care.
   (g) Clean your hands frequently.
   (h) Stay in a separate room from other family members, and if not possible, wear a medical mask.
   (i) Keep the room well ventilated.
   (j) If you share a room, place beds at least 1 meter apart.
   (k) Monitor yourself for any symptoms for 14 days.

2. Call your health care provider immediately if you have any of these danger signs:
   (a) Difficulty breathing
   (b) Loss of speech or mobility
   (c) Confusion
   (d) Chest pain

3. Stay positive by keeping in touch with loved ones by phone or online, and by exercising at home.

REVIEW OF LITERATURE

Recently several probability distributions with different estimation methodologies have been applied on the different data sets of COVID-19 confirmed cases registered in China and in other countries of the world affected from the COVID-19. Some of the studies have been summarized below.

Jung et al., (2020) concluded that the COVID-19 pandemic has significant capability for initiating a pandemic by using binomial and gamma distributions and verified the results by using the method of maximum likelihood estimation. Linton et al., (2020) suggested that based on 95th percentile estimate of the incubation period the minimum duration of isolation must be nine days. They noted the median time delay of 13.8 days from the start of disease to demise must be accounted for when assessing the case casualty threat of this virus by using the lognormal, Waybill, and gamma distributions.

Backer et al., (2020) calculated that the average incubation period was 6.4 days, varying from 2.1 - 11.1 days by using Waybill, gamma and lognormal distributions. Zhang et al., (2020) proposed that the median reproductive number (Ro) of COVID-19 was approximately 2.28 and the 95% confidence interval for this median was 2.06 – 2.52 through the initial stage experienced on the Diamond Princess cruise ship by using gamma and Poisson distributions and they did estimate the values of unknown parameters by the method of likelihood estimation.

Nishiura et al., (2020) concluded that the CIVID-19 serial interval was less than or equal to its median incubation period. That proposes a consider rabble fraction of secondary spread may happen preceding to sickness beginning. The serial interval of COVID-19 is also less than the serial interval of SARS, showing that computations made by using the serial interval of SARS may lead bias. Kucharski et al., (2020) proposed that the transmission of COVID-19 has perhaps deteriorated in Wuhan through end of January 2020, corresponding with the control to travelling. As further cases come to in other countries with like spread probable to Wuhan formerly these regulator measures, may be they will not meet initially, but might leading to new epidemics ultimately.

Lauer et al., (2020) estimated that median incubation period of COVID-19 was almost five days, identical to SARS, the consequences support present suggestions for the quarantine time to COVID-19 is 14 days, by fitting the gamma, Waybill and Erlang distribution on the data of 181 confirmed cases from corona virus in China. Gholami et al., (2020) compared the different probability distributions of confirmed cases and deaths due to COVID-19 recorded in Iran. They concluded that the best-fitted distribution was Waybill in both confirmed cases and deaths as compared to normal and lognormal distributions.

Verity et al., (2020) assessed that mean period from the beginning of indications to demises to be 17.8 days and discharge from hospitals to be 24.7 days. They calculated a best estimate of the case death ratio in China is 1.38% with substantiality higher proportions I senior age groups 0.32% in persons with age less than 60 years vs 6.4% in persons having age 60 years or more .For persons having age 80 years or more the ratio is around 13.4% while the case fatality ratio estimates from worldwide cases stratified by age were reliable with those from China 1.4% in persons having age less than 60 years and for 60 years or more it is 4.5%, by applying Bayesian methods and fitting gamma distributions to data from start to death and start to recovery. Liu et al., (2020) evaluated mean Ro (basic reproduction number) for COVID-19 was about 3.28 having median 2.79 and quartile deviation 1.16 by using stochastic and statistical methods.

2.1 Databases

A systematic review will be performed through seven databases:

(a) MEDLINE (Medical Literature Analysis and Retrieval System Online)
(b) ISI of-Knowledge, CENTRAL (Cochrane Central Register of Controlled Trials)
(c) EMBASE (Excerpt a Medical Database)
(d) SCOPUS
(e) LILACS (Latin American and Caribbean Health Sciences Literature)
(f) PsycINFO (Psychology Information)
(g) CNKI (Chinese National Knowledge Infrastructure) from inception until 30 June 2020.

No restriction regarding the publication date, setting or languages will be considered. Preliminary search strategies were carried out on 29 March 2020 and will be updated in June 2020. The primary outcomes will be the prevalence and the severity of psychological symptoms in young people and adults (>18 years old) resulting from the impact of COVID-19 pandemic. Study selection will follow the Preferred Reporting Items for Systematic Reviews and Meta-Analyses checklist. Pooled standardized mean differences and 95% CIs will be calculated. The risk of bias of the observational studies will be assessed through the Methodological Index for Non-Randomized Studies (MINORS). Additionally, if sufficient data are available, a meta-analysis will be conducted. Heterogeneity between the studies will be determined by the I2 statistics. Subgroup analyses will also be performed. Publication bias will be checked with funnel plots and Egger’s test. Heterogeneity will be explored by random-effects analysis.

2.2 Strengths and Limitations of this Study

This systematic review protocol reduces the possibility of duplication, gives transparency to the methods and processes that will be used, reduces possible biases and allows peer review.

We will provide evidence in order to inform, support and customize shared decision-making from the healthcare providers, stakeholders and governments in this context of global outbreak of the corona virus. This systematic review will be the first to evaluate critically the scientific evidence from the observational studies about the impact of the COVID-19 pandemic on the mental health of young people and adults. The heterogeneity of the studies as well as the methodological appraisal and the probably reduced number of studies (due to the recent COVID-19 outbreak) might be the main limitations of this systematic review.

2.3 Research Aims

The purpose of this systematic review is to critically synthesis the scientific evidence about the impact of the COVID-19 pandemic on the mental health of young people and adults.

2.4 Study Selection

Regarding the study design, we will include only observational studies that investigated the prevalence and the severity of psychological symptoms of young people and adults (>18 years old) resulting from the impact of the COVID-19 pandemic. Nevertheless, studies that analyzed mental and behavioral disorders due to the use of alcohol and other drugs will be excluded. Studies carried out with children, adolescents, pregnant women and the elderly people will be excluded. Randomized controlled trial (RCT), non-randomized controlled trial (NRCT), qualitative studies and the grey literature will also be excluded. This systematic review has no restriction with regard to the languages as well as settings of the target population.

METHODOLOGY

This chapter is about to research methodology and procedure used in this study to discover the research problem. The key purpose of the study is to find the perceptions of students towards impact of COVID-19 pandemic on mental health of young peoples. Researcher aim is to identify the causes & effects of student perceptions toward impact of COVID-19 pandemic on mental health of young peoples and formulate suggestions and guidelines in reducing impact of COVID-19 pandemic on mental health of young people is based on student's perceptions.

3.1 Design of the Study

Primary method of research is use in the research, quantitative approach using questionnaire was use to collect the data about students, perception regarding computer-based examination (CBE) system. This is most appropriate method to investing students, point of view about CBE method. Questionnaire have developed for this purpose, which consists of twenty-five question or statements about CBE method, and four demographic characteristics question i.e. gender, age limit, field of study and study level. A five-point liker scale ranging from (agree disagree, uncertain, disagree, strongly agree, yes, no,) was adopted. This study was done on the students was had already used CBE method.

3.2 Target Population

The target population for this research study includes all the active students of virtual university of Pakistan. All study course (like science, computer science, arts etc.) of and all study level (bachelors, masters, M.Phil. and PhD) of vu students are the part of target population.

Table 3.1
Target Population

Population	Male	Female	Total	Selected Sample
Virtual University of Pakistan (Spring 2021)	48	50	1000	98

3.3 Delimitation of the Study

Keeping in view limited time and access resources with researcher, the study have delimited to

(a) Only students of Virtual University of Pakistan (Spring 2021)
(b) Only education session of spring 2021 was select.
(c) Both male and female students were included in the study, so that perceptions of students regarding impact of COVID-19 pandemic on mental health of young peoples were determined across both the gender.

3.4 Study Sample

Students of size (98) had randomly selected for this study. These students have selected from 1000 male and female students from VU. According to (*Fraenkal & Wallen)*, for a survey study an appropriate representative sample would be 10% of the total population. A total of 100 male and female students from VU (spring 2021) were selected through simple random sampling technique

Questionnaire was send to all the selected students via email, Facebook and Whatsapp. Data was collect in two days.

3.5 Research Instrument

As the study was descriptive in nature and survey technique was used to collect data. A questionnaire was developed comprising 25 items. Researcher developed this questionnaire based on literature, related studies and under the guidance of worthy supervisor.

3.6 Research Ethics

Following research ethics were kept in mind:

(a) Collected information was used for research purpose only.
(b) Participants were informed before filling the questionnaire

3.7 Data Collected Procedure

After all the data was collected, statistical procedure was conducted to analyze the answer of research statements. Statistical procedure involves frequency and percentage, after calculating the frequencies and percentages for response; they are first presented through tables and then visually by using a pie chart graph.

3.8 Data Analysis Technique

Descriptive statistics were calculated which includes frequencies and percentages for data analysis. By using these frequencies and percentages, tables were drawn to show the results numerically and pie chart graphs were drawn to show the results graphically.

DATA ANALYSIS

Descriptive statistics are discussed first, followed by the responses of designed themes. Demographic data of respondents are shown by graph.

4.1 Respondent`s Details

Table 4.1
Respondent`s Details

Gender	Frequency	Percentage %
Male	48	49%
Female	50	51%
Transgender	0	0%
Total	98	100%

Table 4.1 shows that out of 98 respondents 48 of them are males with 49% while 50 of them are females with 51%. The response rate of females is higher than males.

4.1.1 Effect of COVID-19 on Students

Scenario (1)

Are we using precautions the mental health effects of COVID 19 can be avoided

a) Yes
b) No

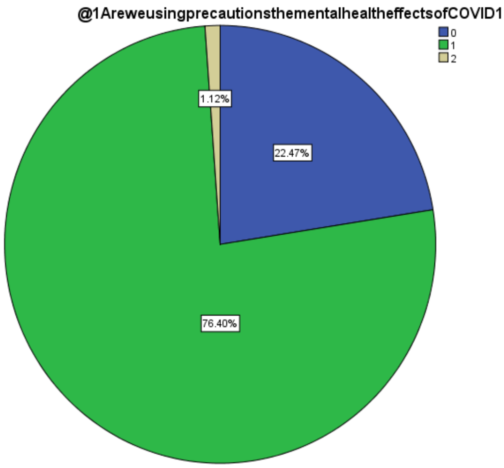

Figure 4.1

Figure 4.1 shows the result through pie chart where we show our data through green and blue color green color cover most of the area its mean they agree that taking precautions increase our mental health. Moreover, skin color cover very small area its mean few respondent do not agree.

Scenario (2)

Do you think that COVID-19 affected the mental health of peoples?

a) Yes
b) No

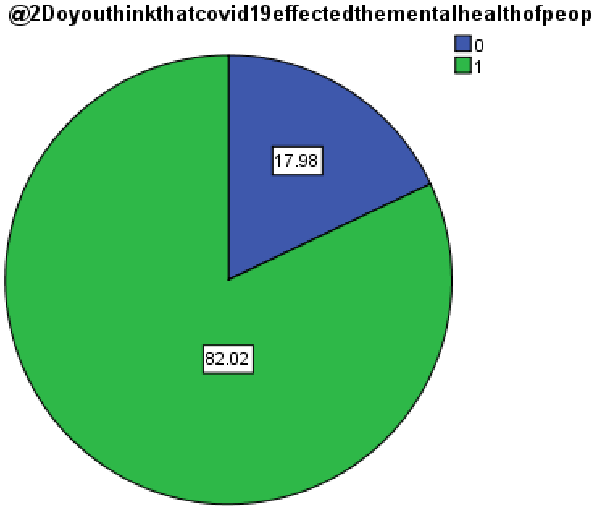

Figure 4.2

In Figure 4.2 Green color show how many people agree with that statement and blue color show how many people disagree with that statement. Green area shows that. Max people think that COVID-19 effect on mental health.

Scenario (3)

At what age you think the COVID has shown its effect more.

a) 15-17
b) 18-20
c) 21-23
d) 24-25

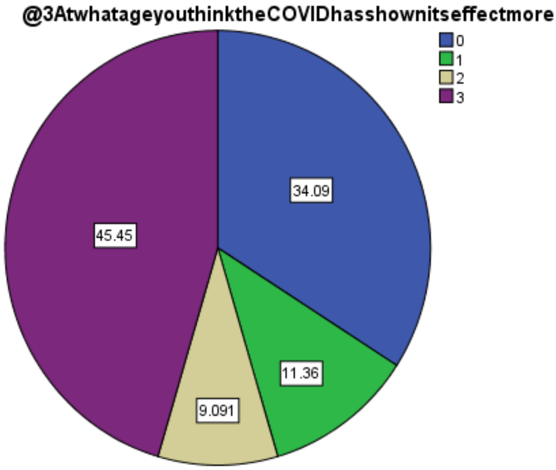

Figure 4.3

Whereas Figure 4.4 shows 4 different color purple, blue, green, and off white. Moreover, green color shows how much people agree that statement and blue color show how many people disagree that statement. Green area shows that Max people think that COVID effect on mental health.

Scenario (4)

How much have COVID-19 effected the human brain?

a) Normal
b) Extremely
c) Very extremely
d) none of these

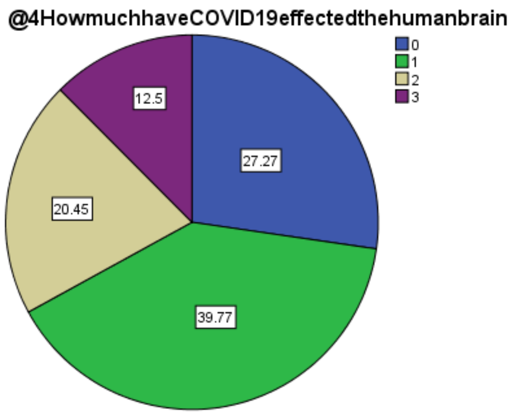

Figure 4.4

Figure 4.4 shows 4 different colors. Skin, blue, purple and green. Where green color cover most of the area of pie chart. Purple color occupies very little space, which represent less people.

How much of the time, during the past 4 weeks, have you felt downhearted or depressed?

a) All
b) Most
c) Some
d) Little
e) None of the time

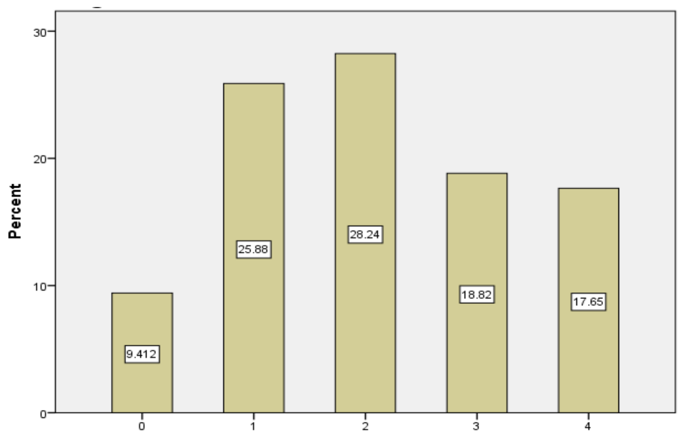

Figure 4.5

Figure 4.5 we can see that there are 5 different bars every bar represent different percentage. Tallest bar with 28.24% cover most of the area that show during COVID people were lonely for some time and smallest bar with 9.412% show less people were lonely all time during COVID-19.

How concerned are you about your health during impacts COVID-19?

a) Not at all
b) Somewhat
c) Very
d) Extremely

Table 4.2

		Frequency	Percent
Valid		17	17.5
	Extremely	1	1.0
	Not at all	30	30.9
	Somewhat	27	27.8
	Very	22	22.7
	Total	97	100.0

In the Table 4.2, we calculate frequency table and percentage. 97 people were asked the question .all of them respond the question. There were four options (extremely, not at all, somewhat and very), most people choose number two option (not at all), its percentage is 30.9 and frequency is 49.5. Most of the people said they did not concern about their lives during COVID-19 just 1% concern about their lives

4.4 Discussion

From the data collected and analyzed in our study. There were findings to discuss here in this section. Our study was done on 98 VU students with male and female distribution was 48 (49%) and 50 (51%) respectively. This showed that female participants were more than male participants were.

Our study analyzed 25 questions regarding the Impact of COVID-19 pandemic on mental health of young peoples. Among 98 students, majority of the participants 70.01% respondent say yes they are agree through precautions we avoided COVID-19 and 20.6 percentage respondent did not agree this statement and 9% respondents did not respond.

Study findings showed that majority of the respondents 73% people agree that COVID-19 affected the mental health of young peoples and 16% did not agree, whereas 11% peoples did not respond. 45.5% respondents in our study were of the view that ages of 20-24 were more effect by COVID-19 and 54.5 said that ages of 15-19 were more affect by COVID-19.

There were 24.7% peoples they said COVID Effected normally, 36.1% peoples they said COVID Effected extremely, 18.6% peoples they said COVID Effected vary extremely whereas 11.6% peoples they said COVID did not effect. There were 57.30% peoples agree with the statement that COVID effected all age of peoples and 42.70 did not agree with the statement. In our study, there were 22.7% peoples they said upper class are more affected 26.2% peoples said middle class were more affected whereas 51.1% peoples said that lower class is more affected. There were 91.01% peoples with "think that the number of deaths due to COVID-19 which reported on social media and TV has affected the mental health of young people's" and 8.9% peoples did not agree with the statement.

Study findings showed that majority of the respondents 77.53% peoples did not agree that government has government taken any action that this disease could affect very few people and 22.47% respondents are agreeing with the statement. According to our study 77.03%, respondents did not agree that lockdown had a negative effect on the minds of young people whereas 26.97% agree with the statement. Study findings showed that majority of the respondents 93.18% didn't agree with the statement that people think who are infected with the corona virus Unemployment has raised due to the downturn that has also had a negative effect on young people's mental health whereas 6.816% agree. In our study there were 49.5% people said no they have-not seen any patient that affected with COVID. Only 40% people agree with this statement the number of those who did not see such patients was high.

There was very important question that did COVID effects on mental health, 81% people agree with that COVID has strongly effects on mental health. Because of COVID-19, people were lock up in their homes and increased unemployment .for these reason led to an increased in mental illness. Only 8% people did not agree, they think that COVID had no effects on their mind 46% people agree with this statement that pandemic effects on social life of young people 46% have seen difficulties in his social life. 8% people said they did not face any problems in this 8% mostly included house women and upper class families. 45% said we endured hardship in this 45% all age group people included of all walks of life and people of all ages 1.163% respondents said COVID-19 had bad effects on the professional life of young people and 98.84% respondents said COVID-19 had not bad effects on the professional life of young people.

Another important question we asked to respondent that are young people endured hardship in supporting their families, 70% people agree with this statement. In this 70% included all type of people like students working people and daily wages people also women faced problems. Just 19% said that they did not face any hardship during supporting their families. There were 7.2% peoples said this pandemic has not changed the thinking behavior of young people, 43.3% said this pandemic has changed the somewhat thinking behavior of young people and 38.1% said this pandemic has changed the very much thinking behavior of young people. There was a question that are you satisfied with your life, 10% fully satisfied with their life 3.1% did not satisfied .daily wages people included in this 3%. 33% people somewhat satisfied with life. Mostly students included this 33%. Student faces educational issues 26% of them face 75% financial problems 20% upper class did not face any financial problems 17% out of 100% face very hard time.

Lockdown create very hard time for lower class people. Due to Lockdown rises unemployment. Lower class faced many difficulties 26% people satisfied with their lives 13% were not satisfied during pandemic. Mostly under 45 to 60 years age group people were not satisfied because of over age 62% respondents give response that they have seen many difficulties by neighbor, friend and relatives. Because many people bound in their houses and workplaces, 24% people did not face any issues.

There were 9.3% peoples said they felt all the time downhearted and depressed, 13.4% peoples said they felt some time downhearted and depressed, 30.5% peoples said they felt most of the time downhearted and depressed. Other peoples said they are not depressed in COVID-19 36% people responded that some time they have been nervous during lockdown because different people face different type of problems. The answer to this question was mostly between the ages of 20 to 60. Just 4% people feel loneliness during all the pandemic condition.

We add also this question in our questionnaires data “how much of the time you have been lonely during COVID because of lockdown condition many people feel loneliness 8% people feel loneliness all the time. Most aged people respondent to this 8% and 24% out of 100 sometime feel loneliness. Over all answered percentage were almost equal. We collect data from 97 people we asked question that how much of the time you have been happy during COVID condition.

According to the data obtained just 5.2 % people said they were happy all the time during COVID But there are also very few people who said they were not happy all that time. There were 30.9% did not agree that they concern about their life during COVID. This question all the people answered percentage almost not equal only 1% respondent extremely concern in lockdown condition. 27.7 % people in some cases they show their concern

4.5 Conclusion

A significant body of studies, analyses, surveys and policy documents already exists on the impact of COVID-19 on young people and the youth sector from multiple perspectives. A first review of the literature shows that education, learning, mobility, unemployment, and mental health are the most researched topics so far. Looking across these studies, key messages emerge on the types of impact and extent to which different sections of the youth population are affected. Yet, more research is needed in order to better qualify the nature and content of the impact, beyond descriptive observations:

(a) Why have some groups of young people been affected more than others?
(b) How different types of impacts are interlinked to each other?
(c) Are new inequalities emerging, beyond the exacerbation of existing ones?

This study is about Impact of COVID-19 Pandemic on Mental Health of Young People and lockdown, aiming at the implementation of a holistic approach that considers both physical and mental health and well-being. The society as a whole, and in particular vulnerable groups such as children, older adults, people with existing mental health disorders, and front-line health-care workers.

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LATENT CLASS TRAJECTORIES OF BIOCHEMICAL PARAMETERS AND THEIR RELATIONSHIP WITH RISK OF MORTALITY AMONG ACUTE ORGANOPHOSPHORUS POISONING PATIENTS: COHORT STUDY

Waqas Ahmed Farooqui[1*], Mudassir Uddin[2], Rashid Qadeer[3] and **Kashif Shafique[1,4]**

[1] School of Public Health, Dow University of Health Sciences Karachi, Pakistan
[2] Department of Statistics, University of Karachi, Karachi, Pakistan
[3] Department of Medicine, Ruth Pfau Civil Hospital Karachi Dow University of Health Sciences, Karachi, Pakistan
[4] Institute of Health and Wellbeing, University of Glasgow Glasgow, United Kingdom
[§] Corresponding author Email: waqas.ahmed@duhs.edu.pk

ABSTRACT

Background: Acute Poisoning is a global public health challenge. Several factors played role in high mortality among acute OP poisoning patients including clinical, vitals, and biochemical properties. The traditional analysis techniques using baseline measurements whereas latent profile analysis is a person-centered approach using repeated measurements. To determine varying biochemical parameters and their relationship with mortality among acute organophosphorus poisoning patients through a latent class trajectory analysis.

Methods: It was a retrospective cohort study had 299 patients' records admitted in between Aug'10 to Sep'16 to Intensive care unit of Civil-Hospital, Karachi. The outcome measure was in-hospital mortality among patients accounting for gender, age, elapse time since poison ingestion, intensive-care-unit stay, and biochemical parameters including random blood sugar, creatinine, urea, and electrolytes (sodium, chloride, potassium). The trajectories of parameters were formed using longitudinal latent profile analysis. These trajectories and repeated measures were used as independent variables to determine and compare the risk of mortality by Cox-Proportional-Hazards models.

Results: A total of 299 patients' data were included with a mean age of 25.4±9.7 years and in-hospital mortality was 13.7%(n=41). In trajectory analysis, patients with high-declining and normal-increasing creatinine, high-remitting, and normal-increasing urea, high-remitting sodium, trajectories observed the highest mortality i.e. 67%(2/3), 75%(6/8), 67%(2/3), 75%(6/8), and 80%(4/5) respectively compared with other trajectories. On multivariable analysis, patients in high-declining creatinine class were sixteen-times [HR:15.7, 95%CI:3.4-71.6], normal-increasing was fifteen times [HR:15.2, 95%CI:4.2-54.6] more likely to die compared with those who had normal consistent creatinine levels. Patients in extremely high-remitting urea trajectory were fifteen-times [HR:15.4, 95%CI:3.4-69.7], normal-increasing urea trajectory was four times [HR:3.9, 95%CI:1.4-11.5] and in high-remitting sodium, the trajectory was six-times [HR:5.6, 95%C.I:2.0-15.8] more likely to die compared with those who were in normal-consistent trajectories of urea and sodium respectively.

Conclusions: Using the latent profile approach, biochemical parameters (creatinine, urea, and sodium levels) were significantly associated with increased risk of mortality among OP poisoning patients.

KEYWORDS

Latent Class; Organophosphorus poison; Biochemical parameters; Repeated Measures

BACKGROUND

Acute Poisoning is a global public health challenge (Farooqui, Uddin, Qadeer and Shafique, 2020). The proportion of suicides due to pesticide self-poisoning varies in low- and middle-income countries between regions, from 0.9% in the European region to 48.3% in the Western Pacific region (Mew et al., 2017). Among other poisonings, the prevalence of OP poisoning vary in neighboring countries (7.7% in Iran (Alizadeh, Hassanian-Moghaddam, Shadnia, Zamani and Mehrpour, 2014), 20.7% in China and India (Dong et al., 2020; Srivastava, Peshin, Kaleekal and Gupta, 2005)). Among all poisoning cases, the prevalence of OP poisoning was stated closely 46.1%, and due to OP poisoning, 2.7% mortality reported in a study from our city (Amir et al., 2019).

Several factors played role in high mortality among acute OP poisoning patients including the age, gender, type, and amount of poison ingested and its biochemical properties, time since ingestion, any pre-existing comorbidities, influence the outcome (Dash, Mohanty, Mohanty and Patnaik, 2008; Hu et al., 2010; E. J. Kang et al., 2009; Singh et al., 2011; Strøm, Thisted, Krantz and Sørensen, 1986). The prognosis of OP depends on the exposure of toxin, the amount of toxin ingestion, and the physiology of compensation. In our country, it is difficult to judge the amount as patients ingest different brands, which lack the description of concentration of the poisonous substance (Amir et al., 2019; Tahir et al., 2014). In clinical settings, the prognosis of these patients is mainly assessed by a variety of different methods including but not limited to vital status, poisoning scoring systems, and laboratory investigations (Ahmed et al., 2016; Anormallikleri, 2010; Bilgin et al., 2005; Coskun et al., 2015; Dayanand & Anikethana, 2015; Dong et al., 2020; C. Kang et al., 2014; Kim et al., 2013; Y. H. Lee et al., 2017; Mohamed, Hasb Elnabi, Moussa, Tawfik and Adly, 2019; Sungurtekin, Gürses and Balci, 2006). Biochemical analysis of blood plays an important role in the diagnosis of intoxicated patients since drugs with biochemical substances produce biochemical changes. Studies have linked biochemical parameters (including amylase, lipase, lactate dehydrogenase (LDH), serum immunoglobulins (IgG, IgA), and creatine phosphokinase (CPK) level) with the severity of OP poisoning but the estimation of these parameters is expensive and most laboratories cannot perform these tests in developing countries.(Bhattacharyya, Phaujdar, Sarkar and Mullick, 2011; Sumathi, Kumar, Shashidhar and Takkalaki, 2014) Therefore, there is a need to identify simple and widely useable biochemical parameters in assessing the severity of poisoning as well as the prognosis of OP poisoning patients (Dayanand & Anikethana, 2015).

Low pseudocholinesterase (PChE), high creatinine (Cr), high sodium (NA^{+}), high blood urea nitrogen (BUN), low Glasgow Coma Scale (GCS) scores, and long hospitalization durations have been assessed for their role in OP poisoning patients' prognosis, but the findings remain inconclusive among OP poisoning patients (Acikalin et al., 2017; Gunduz et al., 2015). One of the main reasons might be the conventional approach of using a single baseline measurement of biochemical parameters to predict the mortality of OP poisoning patients. As these biochemical parameters such as level of, random blood sugar, creatinine, blood urea nitrogen, electrolytes, anticholinesterases, red cell distribution width, lactate dehydrogenase, amylase, creatinine kinase, hematocrit, c-reactive protein are dynamic and tend to change substantially over time and also quite dependent on the physiological response of the patient, which vary significantly from patient to patient.

In such a situation, it might not be a suitable method during the follow-up of linking mortality with single measurements at the time of presentation. Because in those studies at baseline only biochemical inquiries were detected, the significant question is, if parameter mean level changes over time, whether that variation leads to some latent classes that are different than the classes made based on a single baseline measurement of the same variable. It might be disposed to misclassification bias if this single observation method used (Farooqui et al., 2020).

Two of the studies reported biochemical investigations among acute OP poisoning patients used repeated measures, one of the studies discussed the use of home perfusion technique and compare blood glucose level and cholinesterase before and after treatment (Li et al., 2017), while the other study investigated the predictive value of serum acetylcholinesterase levels measured at five different days and its relationship with different neurological syndromes levels (Aygun et al., 2002).

The classical approach to deal with longitudinal repeated-measures (RM) is analysis of variance (RMANOVA) but new forms of Structural Equation Modeling (SEM) provides new approaches for repeated measure designs (McArdle, 2009).

In such a situation, latent growth modeling (LGM) provides a better alternative to observe and estimate growth trajectories overtime for dynamic variables. SEM advances basic longitudinal analysis of data to include latent variable growth over time while modeling both individual and group changes using slopes and intercepts. When these variables are continuous the technique called longitudinal latent class analysis (LLCA) or more specifically longitudinal latent profile analysis (LLPA) (Vasantha & Venkatesan, 2014). The traditional analysis techniques are analysis of variance, multiple regression, and multilevel models which are variable-centered approaches whereas LCGA is a person-centered approach focused on identifying unobserved subpopulations comprising similar individuals (Wang & Wang, 2019). To the best of our knowledge, there is no previous study that has compared the repeated measures and latent trajectories of biochemical parameters in OP poisoning patients and their relationship with mortality.

Therefore, the present study aimed to analyze the growth trajectory of biochemical parameters among OP poisoning patients and comparing two approaches of biochemical parameters individual response (patterns i.e. LLPA vs RM) and their relationship with mortality using survival analysis.

METHODS

Data Source

All OP poisoning patients admitted during August 2010 to September 2016 at medical ICU of a tertiary care hospital from Karachi-Pakistan i.e. Dr. Ruth K.M. Pfau / Civil Hospital were included in this study. This hospital is one of the largest tertiary care hospitals in the province of Sindh - Pakistan with an annual patient turnover of approximately > 4,000,000.

Study Participants

A total of 299 OP poisoning patients' older than 13 years of age and whose biochemical parameter data were available were included in this study. Only those patients were included in this study whose at least two to four observations of biochemical parameters with 12 to 24 hours' interval within 0 to 96 hours of admission were available in medical records.

Study Design and Sample

This was a retrospective cohort study on OP poisoning patients who was shifted to ICU.

Data Collection Tool

A proforma was used to retrieve demographic, clinical characteristics, and biochemical parameters' information.

Ethical Consideration

Ethics approval was obtained from the Institutional Review Board of Dow University of Health Sciences (DUHS) (Ref No. IRB-560/DUHS/Approval/2015/75 dated 11th Jun 2015) and Board of Advanced Studies and Research (BASR) (BASR/No./02505/Sc. dated 2nd October 2015) of the University of Karachi, Pakistan. All methods were carried out in accordance with relevant guidelines and regulations. The Institutional Review Board of DUHS waived the consent due to the retrospective nature.

Study Variables

Each patient's data was obtained from medical records.

Independent variables: Demographics (age, gender), elapse time since poison ingestion and ICU stay. In addition to these variables, biochemical parameters including random blood sugar, creatinine, blood urea nitrogen, sodium electrolyte, chloride electrolyte, potassium electrolyte were also retrieved for an initial period of four days as per completion of medical records. These biochemical parameters were captured overtime during the hospital stay.

Dependent variable: In-hospital mortality data were recorded to assess the outcome of patients within hospital stay.

Statistical Analysis

The data were entered and managed using MICROSOFT OFFICE 365 EXCEL. LLPA was utilized for identifying unknown latent classes of individuals with respect to measurements at different time points. It does not model changes over time, but the patterns of measures that span multiple time points (Wang & Wang, 2019). LLPA used in MPLUS version 7.0 software to detect patterns of biochemical parameters. Models were prepared as per guidelines (Muthén & Muthén, 2009; Wang & Wang, 2019). Our models used first-order polynomials observed linear growth pattern and for each class (two/three/four) whichever best on fit indices to find the maximum latent classes. For every individual, we computed the posterior probabilities and latent class. We choose classes based on each latent class proportion size of at least 5 cases (1.5%) and best-fitting indices like Log-likelihood, Lo-Mendell-Rubin Test, Bootstrap Likelihood-Ratio Test, Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), sample size adjusted BIC (Tein, Coxe and Cham, 2013).

Based on the modeled graphical shapes, labels were assigned to the trajectories (Figure 1). These were based on slope and intercept of mean of individual's pattern. Based on the start of biochemical parameters ranges, we defined labels first part (e.g. low, normal, high, extreme) and on their patterns of repeated measure we identified second part (e.g. consistent, stable, declining, increasing, remitting (up then down or vice versa)). Form of multiple repetition of high and low observations refers to the term remitting during the course of observations i.e. 4 days starting from high to low or vice versa (Farooqui et al., 2020).

For inferential analysis and figures used STATA version 15.0 software. To detect differences in age, total ICU stays and baseline biochemical parameters between dead and alive patients, Wilcoxon rank-sum test was applied. Association between gender and biochemical parameters latent classes with mortality were assessed using chi-square and Fisher exact test where appropriate.

For survival analysis, total ICU stay time was used to define risk time. We computed unadjusted and adjusted hazard ratios (HR) and respective confidence intervals for mortality and at the same time compared two approaches using independent variables as assigned trajectory and, in another approach taking independent variables as repeated measures. Cox proportional hazards model was applied. In multivariable analysis, we adjusted for age and approximate time elapsed since ingestion of poison. We evaluated the proportional hazards assumption of Schoenfeld residuals by phtest and power of the Cox Model using Harrell's C concordance statistic; assumptions were identified satisfactory and not violated (Cleves, Gould, Gould, Gutierrez and Marchenko, 2008; Mirza et al., 2016). P-values < 0.05 were considered statistically significant.

RESULTS

A total of 499 OP poisoning patients (of either gender) were eligible during the six months data collection period from June 2016 to November 2016. Out of the total, 200 patients' records were excluded due to incomplete data, wrong registration, self-reported history of chronic conditions such as hypertension, diabetes mellitus, osteoarthritis, asthma, or pregnant women, less than 3 days ICU stay. A total of 299 patients' data was included for final analysis with a mean ± standard deviation age of cohort 25.4 ± 9.7 years (ranged 13-70 years) with an overall mortality of 13.7% (n=41). Biochemical parameters (including random blood sugar, creatinine, blood urea nitrogen), age, elapsed time since ingestion of poison were significantly higher among dead as compared to alive individuals, while gender, total ICU stay, and electrolytes were not significantly different between dead and alive individuals (Table 1).

Table 1
Descriptive Statistics of Baseline Characteristics with Mortality

Characteristics	**Alive (N=258, 86.3%)**	**Dead (N = 41, 13.7%)**	**P-value**
Gender			
Female	131 (87.3)	19 (12.7)	0.598˜
Male	127 (85.1)	22 (14.9)	
Age (years)	23 (18 – 28)	27 (20–40)	0.009□
Elapse Tie (hours)	5.2 (2.5 – 8.4)	10 (8.0–12.5)	< 0.001□
ICU Stay (days)	5.7 (3.8 – 10.8)	7.5 (4.1–10.5)	0.519□
Biochemical			
RBS (mg/dl)	113.0 (100.0–151.0)	143.0 (105.0–185.0)	0.018□
Cr (mg/dl)	0.7 (0.6–0.9)	0.9 (0.7–1.1)	0.009□
BUN (mg/dl)	10 (7–12)	11 (9–15)	0.012□
Electrolytes			
Sodium (Na^+) (mEq/L)	141 (138–143)	140 (137–143)	0.600□
Chloride (Cl^-) (mEq/L)	105 (102–108)	104 (101–107.5)	0.555□
Potassium (K^+) (mEq/L)	3.9 (3.5–4.1)	3.9 (3.5–4.1)	0.564□

IQR, interquartile range (25th–75th percentile); □Wilcoxon Rank Sum Test;
˜Chi-square Test; ICU: intensive care unit; RBS: Random Blood Sugar;
Cr: Creatinine; Blood Urea Nitrogen

Three distinct trajectories of creatinine (normal-consistent, high-declining and normal-increasing) were identified among 299 OP poisoning patients (Figure 1), mortality in normal-consistent trajectory was 36/292=12.3%, in normal-increasing was 03/04=75.0%, while mortality in high-declining trajectory was 02/03=66.7%) (Table 2).

Four trajectories of urea were identified (normal-consistent, high-declining, extreme high remitting, and normal-increasing) (Figure 1). Patients' mortality in normal-consistent trajectory was 29/279=10.4%, in high-declining was 04/09=44.4%, in extremely high-remitting was 02/03=66.7%, while mortality in normal-increasing trajectory was 06/08=75.0%) (Table 2).

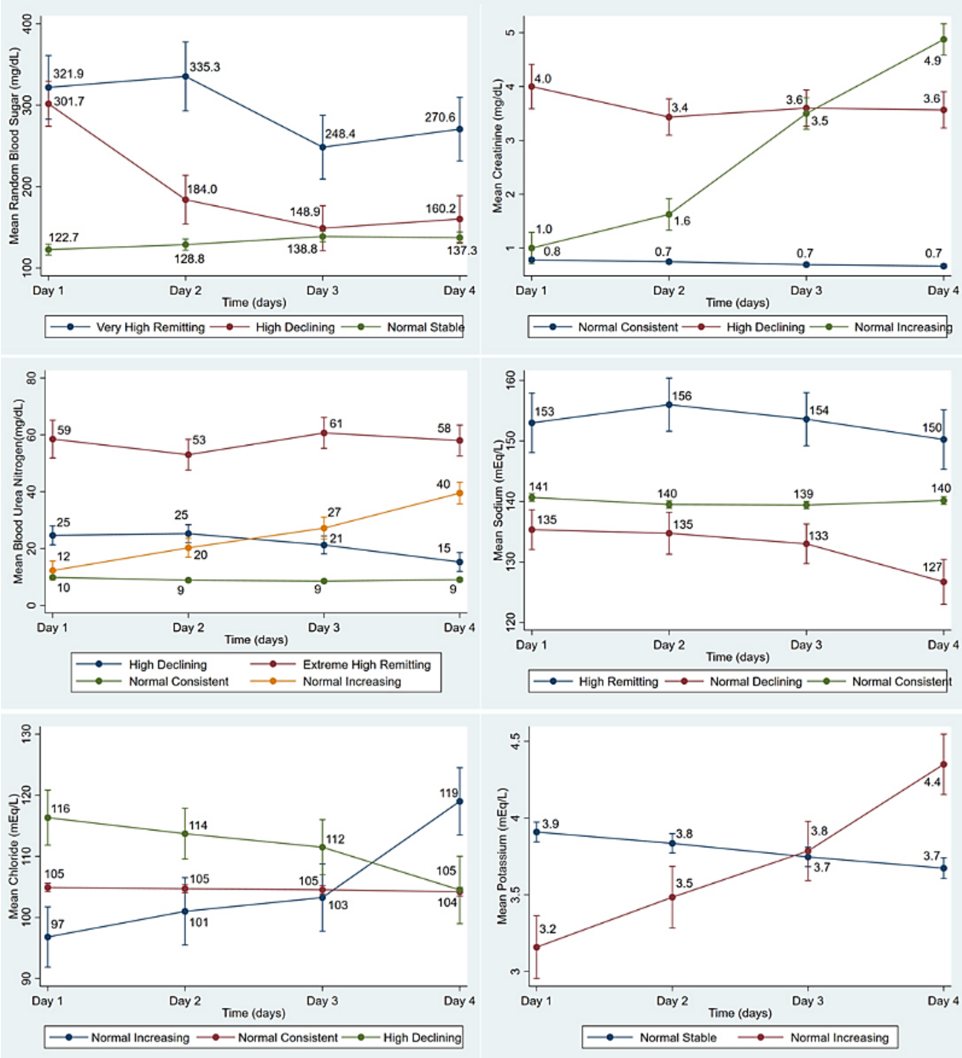

Figure 1: Trajectories of Biochemical Parameters from day 1 to 4

Three trajectories were identified for sodium electrolyte (normal-consistent, high-remitting, and normal declining) (Figure 1). Patients' mortality in normal-consistent trajectory was 37/285=13.0%, in extremely high-remitting it was 04/05=80.0%, while mortality in normal declining trajectory was 0/9=0% (Table 2).

Table 2
Mortality Comparison of Biochemical Parameters - Latent Classes

Parameters	Total (N = 299)	Alive N = 258 (%)	Dead N = 41 (%)	P-value^
RBS				
Normal Stable	278	243 (87.4)	35 (12.6)	
Very High Remitting	07	05 (71.4)	02 (28.6)	0.070
High Declining	14	10 (71.4)	04 (28.6)	
Creatinine				
Normal Consistent	292	256 (87.7)	36 (12.3)	
High Declining	03	01 (33.3)	02 (66.7)	0.001
Normal Increasing	04	01 (25.0)	03 (75.0)	
Blood Urea Nitrogen				
Normal Consistent	279	250 (89.6)	29 (10.4)	
High Declining	09	05 (55.6)	04 (44.4)	<0.001
Extremely High Remitting	03	01 (33.3)	02 (66.7)	
Normal Increasing	08	02 (25.0)	06 (75.0)	
Sodium				
Normal Consistent	285	248 (87.0)	37 (13.0)	
High Remitting	05	01 (20.0)	04 (80.0)	0.002
Normal Declining	09	9 (100.0)	0 (0)	
Chloride				
Normal Consistent	287	250 (87.1)	37 (12.9)	
Normal Increasing	05	03 (60.0)	02 (40.0)	0.073
High Declining	07	05 (71.4)	02 (28.6)	
Potassium				
Normal Stable	272	236 (86.8)	36 (13.2)	0.392
Normal Increasing	27	22 (81.5)	05 (18.5)	

^Fisher Exact Test

On multivariable analysis after adjusting for age, when modeled repeated measures, creatinine increased the risk of mortality [HR:1.17, 95% CI:1.12 – 1.22, P-value < 0.001]. However, on multivariable analysis after adjusting for age and elapsed time when modeled latent classes, patients with high-declining creatinine were sixteen times more likely [HR:15.7, 95% CI: 3.4 - 71.6, P-value < 0.001] and normal-increasing were fifteen times more likely [HR:15.2, 95% CI: 4.2 - 54.6, P-value < 0.001] to die compared with those who were in normal-consistent trajectory. In the multivariable model, repeated measures analysis, urea showed a significant relation with mortality [HR: 1.01, 95% CI: 1.01 – 1.02, P-value < 0.001]. However, while using latent classes, patients in high-remitting urea trajectory were fifteen times [HR: 15.4, 95% CI: 3.4 - 69.7, P-value < 0.001] and in normal-increasing urea trajectory were four times [HR: 3.9, 95% CI: 1.4 - 11.5, P-value = 0.012] more likely to die compared with those who had normal-consistent urea. In multivariable model repeated measures analysis, sodium did not show increased risk of mortality [HR: 1.0, 95% CI: 0.999 – 1.001, P-value = 0.958]. However, on latent classes, patients in high-remitting sodium trajectory were six times [HR: 5.6, 95% CI: 2.0 - 15.8, P-value = 0.001] more likely to die compared with those who had normal consistent sodium (Table 3).

Table 3
Two Approaches in Relationship of Mortality with Biochemical Parameters using Cox Model

Parameters	Unadjusted HR (95% C.I)	Adjusted HR(95% C.I)
Repeated Measures Approach		
Random Blood Sugar (mg/dl)	1.0 (0.999 , 1.001)	1.0 (0.999 , 1.001)
Creatinine (mg/dl)	1.18 (1.13 , 1.22)	1.17 (1.12 , 1.22)
Blood Urea Nitrogen (mg/dl)	1.02 (1.01 , 1.02)	1.01 (1.01 , 1.02)
Electrolytes – Sodium(mEq/L)	1.0 (0.997 , 1.002)	1.0 (0.999 , 1.001)
Electrolytes – Chloride(mEq/L)	1.0 (0.996 , 1.003)	1.0 (0.997 , 1.001)
Electrolytes - Potassium (mEq/L)	1.01 (0.93 , 1.11)	1.01 (0.98 , 1.05)
Latent Class Approach		
RBS		
Normal Stable	1.0	1.0
Very High Remitting	2.1 (0.5 , 8.6)	2.9 (0.7 , 12.3)
High Declining	1.9 (0.7 , 5.4)	1.0 (0.3 , 3.2)
Creatinine		
Normal Consistent	1.0	1.0
High Declining	11.6 (2.7 , 49.6)	15.7 (3.4 , 71.6)
Normal Increasing	17.5 (5.1 , 60.5)	15.2 (4.2 , 54.6)
Blood Urea Nitrogen		
Normal Consistent	1.0	1.0
High Declining	3.5 (1.2 , 10)	2.4 (0.8 , 7.0)
Extremely High Remitting	13.1 (3.0 , 55.9)	15.4 (3.4 , 69.7)
Normal Increasing	8.2 (3.3 , 20.4)	3.9 (1.4 , 11.5)
Electrolyte - Sodium		
Normal Consistent	1.0	1.0
High Remitting	6.2 (2.2 , 17.4)	5.6 (2.0 , 15.8)
Normal Declining	-	-
Electrolyte - Chloride		
Normal Consistent	1.0	1.0
Normal Increasing	2.0 (0.4 , 11.0)	2.4 (0.4 , 12.8)
High Declining	2.3 (0.5 , 9.5)	2.7 (0.6 , 11.9)
Electrolyte - Potassium		
Normal Stable	1.0	1.0
Normal Increasing	0.9 (0.3 , 2.3)	0.88 (0.46 , 1.68)

HR: Hazard Ratio, Adjusted Covariates: (For Repeated Measure: Age, For Latent Class: Age and Elapse Time)

DISCUSSION

Summary of Findings

Among OP poisoning cases, higher age and longer elapsed time since ingestion were significantly associated with mortality. Patients in high-declining and in normal-increasing creatinine trajectories and those who were in extremely high-remitting and normal-increasing blood urea trajectories, had significantly high mortality compared to patients in normal consistent creatinine trajectory and urea trajectory respectively. Additionally, high-remitting sodium trajectory has significant mortality compared to individuals in normal consistent sodium level trajectory.

The mortality in our study is comparable to previously published papers on OP poisoning from different regions including neighbor countries (Ahmed et al., 2016; Amin, Abaza, El Azawy and Ahmed, 2018; Amir et al., 2019; Dong et al., 2020; Gunduz et al., 2015; Majidi et al., 2018; Mundhe, Birajdar and Chavan, 2017), however our study appeared to show higher (13.7%) mortality compared to another recently published paper (2.7%) from National Poisoning Control Centre of our urban city. Low mortality in the previous study was perhaps because the study had only six months records available (46.1% (1174/2546)) for those OP poisoning patients who were managed in medical wards(Amir et al., 2019) whereas our study spanned over six years patients data recorded from OP poisoning patients admitted in ICU. Longer time elapsed since ingestion of poison and higher age was significant predictors of mortality and findings are quite consistent with previously published studies (Ahmed et al., 2016; Amin et al., 2018; Gunduz et al., 2015).

Trajectory analysis showed that two trajectories of creatinine (including high-declining and normal increasing) had a significant relationship with mortality among OP poisoning patients. These results were consistent with previous studies as high creatinine found a significant factor of mortality in multiple Asian studies from 2013 to 2018 including one of our urban cities (Ahmed et al., 2016; Gunduz et al., 2015; Kim et al., 2013; J. H. Lee et al., 2013; Y. H. Lee et al., 2017). Two of Asian studies reported risk of mortality on univariable analysis, patients having high creatinine level founds seven to eight times more likely to die (C. Kang et al., 2014; S. B. Lee et al., 2018), whereas in our study patients in normal-increasing trajectory were eighteen times and high-declining were twelve times more likely to die compared to the normal consistent creatinine level. This may be due to the mortality within 30 days in these studies whereas in our study it was 20 days.

Trajectory analysis also shown that two trajectories of blood urea nitrogen (including extremely high-remitting and normal increasing) were significantly associated with mortality among OP poisoning patients. Our study showed four times and fifteen times increased risk of mortality for normal increasing and extremely high remitting trajectories patients respectively, when compared with normal consistent urea trajectory. These results were somewhat supported by two Asian studies which shown high urea level was associated with mortality (Kim et al., 2013; J. H. Lee et al., 2013), and were contrasting with few European and Asian studies which shown urea was not a significant factor of mortality (Ahmed et al., 2016; Gunduz et al., 2015; Y. H. Lee et al., 2017). The LLPA shown in our study that high-remitting trajectory of sodium electrolyte was significantly associated with increased mortality among OP poisoning patients, however previous evidence shown no relationship between sodium and mortality among OP poisoning patients (Ahmed et al., 2016; Gunduz et al., 2015). The difference of urea and sodium may be due to data collection from the emergency department and a single baseline reading in these studies. The results of random blood sugar from our analysis and previous evidence remains consistent showing no relationship with mortality (Ahmed et al., 2016; Gunduz et al., 2015; Li et al., 2017).

In a comparison of the two techniques repeated measures and latent trajectories, on multivariable analysis, when modeled repeated measure creatinine, and urea level showed a significant relationship with low risk of mortality, while sodium level does not increase the risk of mortality. However, when modeled latent classes patients in high-declining creatinine class were sixteen times, normal-increasing was fifteen times more likely to die compared with those who had normal-consistent creatinine level. Patients in extremely high-remitting urea trajectory was fifteen times, normal-increasing urea trajectory were four times and in high-remitting sodium trajectory were six times more likely to die compared with those who were in normal-consistent trajectories of urea and sodium respectively. This reveals that repeated measure approach is predicted low risk of mortality using creatinine and urea levels while latent class trajectories predicted a high risk of mortality using creatinine, urea, and sodium levels.

Previous studies only included baseline measurements of biochemical parameters and examined their relationship with mortality, while this study was unique in the sense that it accounted for multiple observations of biochemical parameters in the first four days of poisoning and linked them with mortality. Our approach is also closer to real-life scenario, where such OP poisoning patients have varying levels of these parameters, therefore the findings can have more application in clinical settings. This study has examined more accessible, relatively inexpensive biochemical parameters and shown more clinical utility and are much more convenient than expensive laboratory-based markers and may be considered for use in clinical settings for future OP poisoning patients admitted for intensive care. Commonly used scoring systems heavily depends on clinical and laboratory investigational information (Davies, Eddleston and Buckley, 2008; Dong et al., 2020; Kim et al., 2013; Mohamed et al., 2019; Persson, Sjoberg, Haines and Pronczuk de Garbino, 1998; Sam et al., 2009; Senanayake, de Silva and Karalliedde, 1993; Wu, Xie, Cheng and Guan, 2016), and rely on single measurements usually at the time of admission of patient. LLPA used in this study is a person-centered approach where latent classes identify trajectories of patients based on repeated measures of very routine clinical parameters (McArdle, 2009).

Strengths and Weakness

This study included multiple observations of biochemical parameters in the first four days of patients' admission into ICU and determined the relationship of different trajectories of biochemical parameters with mortality among OP poisoning patients. To the best of our knowledge, this is the first study that determined the relationship of biochemical parameters with mortality among OP poisoning patients using latent profile approach. To the best of our knowledge, this is the first study that compared the two statistical approaches i.e. latent profile research and repeated measures analysis to determine the risk of mortality among OP poisoning patients. The current study had a reasonably larger sample size by including data of several years from ICU of a tertiary care hospital, which provided generalizable results in our context.

The current study had an estimated amount of poison ingested from medical records, which might not have been very accurate in terms of the actual amount ingested by the patient. Our study was only limited to routinely performed biochemical analysis among OP poisoning patients and could not include arterial blood gas such as base deficit and other biomarkers such as anticholinesterases, lactate dehydrogenase because the repeated measures of these markers were not available in medical records for trajectory analysis.

CONCLUSION

Our study shows that the latent classes of biochemical parameters, high-declining and normal-increasing trajectories for creatinine, extremely high-remitting and normal-increasing trajectories of blood urea nitrogen, and high-remitting trajectory of sodium electrolyte are significant predictors of mortality among acute OP poisoning patients. The latent profile technique appeared to provide better results and prediction compared to conventional repeated measure analysis for such OP poisoning patients.

LIST OF ABBREVIATIONS

OP: Organophosphorus; LLCA: longitudinal latent class analysis; LLPA: longitudinal latent profile analysis; LCGA: Latent class growth analysis; BUN: blood urea nitrogen; PChE: Pseudocholinesterase; Cr: Creatinine; NA^{+}: Sodium; RBS: Random blood sugar; IQR: Inter-quartile range; ICU: Intensive care unit; GCS: Glasgow Coma Scale; AIC: Akaike information criteria; BIC: Bayesian information criteria; HR: Hazard ratio; RMANOVA: Repeated-measures analysis of variance; SEM: Structural equation models.

DECLARATIONS

Ethics approval and consent to participate

The study was approved by the Institutional Review Board of Dow University of Health Sciences (DUHS) since teaching Hospital Civil is affiliated with DUHS. The consent form was not applicable as the study design was retrospective.

Consent to Publish

Not applicable

Availability of Data and Materials

Data included in the current study are not publicly available to ensure confidentiality of the patients but are available from the corresponding author on reasonable request.

Competing Interests

The authors declare that they have no competing interests.

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Authors' Contribution

KS and WAF conceived the idea, draft of manuscript and result interpretation, RQ provided clinical perspective, provided access to data, MU helped in literature review, modeling and referencing, WAF performed compilation, transformation, and statistical analysis. All authors read and approved the final manuscript.

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